Vertmont State Mathematics Coalition Talent Search October 1, 2008

Test 1 of the 2008 – 2009 school year

Problem 1.
Find all the points of intersection of the surfaces whose equations are as follows:

$$z^2 = 2xy - 100 \quad \text{and} \quad z = \frac{x^2 + 2y^2 - 100}{2y}$$

Solution:
Rewrite the equations as $2xy - z^2 = 100$ and $x^2 + 2y^2 - 2yz = 100$.
Subtract the first equation from the second to obtain

$$0 = x^2 - 2xy + y^2 + y^2 - 2yz + z^2 = (x - y)^2 + (y - z)^2$$

The only way the sum of two squares can be zero is if each term is 0; hence $x = y = z$.
Substituting into either equation yields $x^2 = 100$.
Thus, there are two points (10,10,10) and (-10,-10,-10).

Problem 2. The Vermont Tennis Club invites 64 players of equal ability to compete in a single elimination tournament (a player losing a match is eliminated). What is the probability that a particular pair of players (for example, Mike and Bob) will play each other at some point during the tournament?
Solution:
There are several ways to solve this problem. The easiest may be to realize there are 63 = 64 -1 matches in all since each match eliminates one player. The number of possible pairings of the 64 players is $\binom{64}{2}$ = (64)(63)/2 , which is the number of ways to choose two items out a set of 64 distinguishable items. All pairings are equally likely and are never repeated. So the probability of a particular pairing occurring during the tournament is just their quotient, namely 63 divided by (64)(63)/2 or 1/32.
Alternatively since all players are interchangeable, we may compute the answer by finding the expected number of players Bob will play against and then divide by 63. Bob has a probability of 1 of playing in the first round, and his probability of making it to each subsequent round decreases by a factor of 1/2 . Since there are 6 rounds, the expected number of players he will play against is therefore equal to
1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 63/32 , and therefore the answer is 1/32 .

Problem 3. Find two 2 digit integers such that the greater is 3 more than three times the smaller, and their sum is the reverse of the smaller number.
Solution:
Let x = the larger number and $10t + u$ the smaller number where t is the tens digit and u is the units digit of the smaller number. Then

$$x = 3(10t + u) + 3$$

$$x + (10t + u) = (10u + t) \text{ or } x = 9u - 9t$$

Therefore

$$30t + 3u + 3 = 9u - 9t$$
$$10t + u + 1 = 3u - 3t$$

and $\quad 13t+1=2u$

So $t=1$, $u=7$, so smaller number is 17 and larger is $3(17)+3=54$.

Problem 4. Find *all* ordered pairs (a,b) which satisfy $5y^2+2xy-80=0$ such that (a,b) are both integers.

Solution:

Write equation as $x=\frac{40}{y}-\frac{5y}{2}$. Then select integer *y* values such that *x* is an integer.

$x(y=8)=5-20=-15$ OK

$x(y=-8)=-5+20=15$ OK

etc….

Total number of (x,y) pairs are:

$(15,-8)$ $(-15,8)$, $(15,2)$, $(-15,-2)$, $(-21,10)$, $(21,-10)$, $(-99,40)$, $(99,-40)$, $(-48,20)$, $(48,-20)$, $(0,4)$, $(0,-4)$

Alternatively, rewrite equation as y(5y+2x) = 80, so *y* is a divisor of 80. Moreover*, y* must be even, for if *y* were odd then both terms in the product would be odd, which is clearly impossible. Additionally, *y* cannot be divisible by 16, since this would force *x* to be nonintegral ; but all other choices of *y* yield integral values for *x*. Hence $y\in\{2,4,8,10,20,40,-2,-4,-8,-10,-20,-40\}$ yielding the same pairs as above.

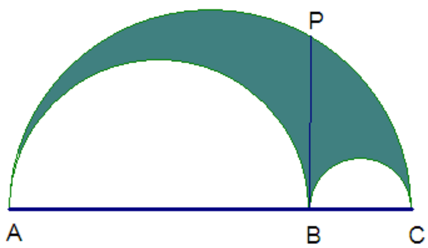
Problem 5. A right triangle has legs of length $\sqrt{22}$ and ($6+2\sqrt{10}$), find the *simplest possible* expression for the length of the hypotenuse. *Hint: Your answer should be a sum of whole multiples of square roots of whole numbers.*

Solution:

The hypotenuse of the triangle is $\sqrt{22+\left(6+2\sqrt{10}\right)^2}=\sqrt{98+24\sqrt{10}}$.

To simplify the nested square root, we guess that $98+24\sqrt{10}$ is either the square of ($a+b\sqrt{10}$) or ($c\sqrt{5}+d\sqrt{2}$). Algebra works out with whole number solutions only for the latter case to give $c=4$ and $d=3$ so solution is $4\sqrt{5}+3\sqrt{2}$. Alternatively, simplify the nested square root assuming $98+24\sqrt{10}$ is the square of $\left(\sqrt{a}+\sqrt{b}\right)$, then $ab=1440$ and $a+b=98$. Then *a,b* are the roots of the quadratic $x^2-98x+1440=(x-18)(x-80)$. Thus you conclude the $\sqrt{98+2\sqrt{10}}=\sqrt{18}+\sqrt{80}=3\sqrt{2}+4\sqrt{5}$.

Problem 6. In the following diagram, line segment **AC** is the diameter of a semicircle. For an arbitrary point B on segment **AC,** construct semicircles with segments **AB** and **BC** as diameters. Let ***h*** denote the length of segment **PB**, where P is a point on the original semicircle with segment **PB** perpendicular to segment **AC**. Express the area of the shaded region in terms of ***h*.**



Solution:
In the diagram, ΔAPC is a right triangle ($\angle APC$ is inscribed in a semicircle) and ΔAPB and ΔPBC are similar to it. Let *d1* and *d2* be the lengths of segments AB and BC, respectively. These are diameters of the smaller semicircles. (For later reference, let ***r1*** and ***r2*** be the corresponding radii.) By similar triangles $\frac{d_1}{h} = \frac{h}{d_2}$ where ***h*** is the length of segment PB. $\therefore h^2 = d_1 d_2$. Now compute the area of the shaded region by subtracting the two smaller semicircular areas from the large one.

$$A = \frac{1}{2}\pi\left[\left(r_1 + r_2\right)^2 - r_1^2 - r_2^2\right]$$
$$A = \pi r_1 r_2$$
$$A = \pi d_1 d_2 / 4$$
$$A = \pi h^2 / 4$$

Problem 7. The first term of an arithmetic progression (AP) is -1 and the 8th term of a geometric progression (GP) is 80. The 4th terms of both progressions are equal. If the 6th term of the GP is equal to the sum of the 6th and 7th terms of the AP, find the smallest possible integral value of the 9th term of the GP.

Solution:
Represent the terms of the progressions as follows:

	1	2	3	4	5	6	7	8	9
AP:	*a-3d*	*a-2d*	*a-d*	*a*	*a+d*	*a+2d*	*a+3d*	*a+4d*	*a+5d*
GP:				*a*	*ar*	ar^2	ar^3	ar^4	ar^5

Note that the 4th term of both progressions are equal. Since the 1st term of the AP is -1 $a - 3d = -1$ and $d = \frac{a+1}{3}$. Since the 8th term of the GP is 80 $ar^4 = 80$ and $r^2 = \sqrt{80/a}$.
Since the sum of the 6th and 7th terms of the AP equal the 6th term of the GP, $(a+2d)+(a+3d) = ar^2$ and $2a+5d = ar^2$. Eliminating *d* results in:
$2a + 5(\frac{a+1}{3}) = a\sqrt{80/a}$ and $\frac{11a+5}{3a} = \sqrt{80/a}$; squaring: $\frac{121a^2+110a+25}{9a^2} = \frac{80}{a}$
and $121a^2 - 610a + 25 = 0$ or $a = 5, a = 5/121$
If $a = 5/121$, the 9th term is not an integer.
If $a = 5$, $r^2 = \sqrt{16}$ and $r = \pm 2$. $\therefore ar^5 = \pm 160$ and smallest 9th term is -160.

Alternatively, view the progressions with common difference d and common ratio r as:

	1	2	3	4	5	6	7	8	9
AP:	-1	-1+d	-1+2d	-1+3d	-1+4d	-1+5d	-1+6d	-1+7d	-1+8d
GP:	$80r^{-7}$	$80r^{-6}$	$80r^{-5}$	$80r^{-4}$	$80r^{-3}$	$80r^{-2}$	$80r^{-1}$	$80r^{-0}$	$80r^{1}$

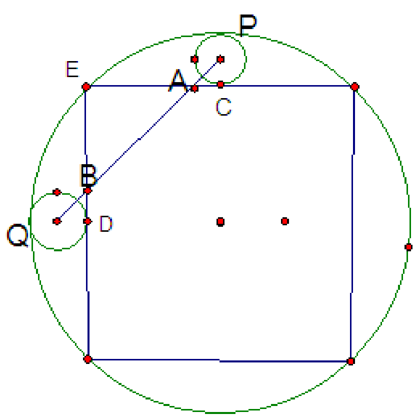
Then the given information implies $(-1+3d)=80r^{-4}$ and $(-2+11d)=80r^{-2}$.

Thus, $80(-1+3d)=80^2r^{-4}=(-2+11d)^2$ so that $-80+240d=4-44d+121d^2$

Or $121d^2-284d+84=0$. Factoring yields $(121d-42)(d-2)=0$ so that $d=42/121$ and $d=2$. The former yields $r^{-2}=(-2+11d)/80=1/44$ which gives the 9th term of the GP as an irrational $\pm160\sqrt{11}$; the later gives $r^{-2}=(-2+11d)/80=1/4$ or $r=\pm2$ which gives the 9th term of the GP as ±160. The smallest integral value is therefore -160.

Problem 8. A square is inscribed in a circle of radius 1. Circles P and Q are the largest circles which can be inscribed in the indicated segments of the circle. The line joining the centers of circles P and Q intersects the square at points A and B. Compute the length of AB.

Solution:



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Let O be the center of the large circle, and C be the point of tangency of P to the square, and D be the point of tangency of Q to the square. Since the radius of circle O is 1,

$OC=OD=\frac{\sqrt{2}}{2}$ which implies that $QD=PC=\frac{1}{2}(1-OC)=\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)=\frac{2-\sqrt{2}}{4}$.

Hence $OQ=OP=PC+OC=\frac{2-\sqrt{2}}{4}+\frac{\sqrt{2}}{2}=\frac{2+\sqrt{2}}{4}$. Since angle QOP is a right angle, by symmetry it follows that $PQ=OP\sqrt{2}=\frac{1+\sqrt{2}}{2}$. Also since ΔPCA and ΔQDB are isosceles right triangles (similar to ΔPOQ) it follows that $QB=PA=PC\sqrt{2}=\frac{\sqrt{2}-1}{2}$

′ The Math Coalition is grateful to contributors for this test including Middlebury College professors Michael Olinick, Bill Peterson , Peter Schumer and Frank Swenton. Also contributing is Tony Trono, retired Burlington High School math teacher and Evan Dummit.