Vermont Talent Search **November 2, 2009**
School Year 2009-2010
Test 1 Solutions

Problem 1.

The fraction $\frac{2}{x^2-3x+2}$ can be written as an infinite series. Find the sum of the first four terms of the series expansion for $x=-1$ and $x=-2$.

Solution:

Using long division, the expansion of $\frac{2}{x^2-3x+2}$ becomes

$$\frac{2}{x^2}+\frac{6}{x^3}+\frac{14}{x^4}+\frac{30}{x^5}+\ldots=\frac{2}{x^2}\left(1+\frac{3}{x}+\frac{7}{x^2}+\frac{15}{x^3}+\ldots\right)$$

Thus for $x=-1$, the first four terms are $2-6+14-30=-20$

And for $x=-2$, the first four terms are $\frac{2}{4}\left(1-\frac{3}{2}+\frac{7}{4}-\frac{15}{8}\right)=\frac{1}{2}\left(\frac{8-12+14-15}{8}\right)=\frac{-5}{16}$

Problen 2.

Many different seven digit integers can be formed by permuting the digits 1- 7; examples include 1236574 and 7531246. Suppose we list every possible seven digit number of this form and then add up all these numbers. What is the sum?

Solution:

There are 7! = 5040 of these numbers. The average of these numbers is clearly 4444444. Thus the total sum is 7! * 4444444 = 22,399,997,760.

Alternatively, in each column of the sum, there are 6! = 720 1's, 2's, 3's, etc.

Thus the sum of each column is 720(1 + 2 + 3 + 4 + 5 + 6 + 7) = 720*28 = 20160. You can then do ordinary addition with carries to get the sum. That is

$$20160\left(10^0+10^1+10^2+\ldots+10^6\right)=22{,}399{,}997{,}760$$

Problem 3.

Two sides of an isosceles triangle have lengths 181 and 38. What is the area of the triangle?

Solution:

Since the sum of the lengths of any two sides of the triangle must exceed the length of the third side, it can not be the case that the two equal sides have length 38. Thus both equal sides have length 181 and the base is 38. The altitude *h* satisfies, by Pythagoras,

$$h^2=181^2-19^2=(181-19)(181+19)=(162)(200)=(81)(4)(100)$$

so h = (9)(2)(10) = 180 and the area is (180)(19) = 3420.

Problem 4.

In baseball's World Series, two teams play games against each other until one of them has won four games. (Thus the series must end no later than the seventh game.) Suppose that the teams are perfectly matched, so that each has a 50% chance of winning any game, and successive games are independent.

Under these assumptions, is it more likely the World Series will end in six games or in seven games? *Explain your answer. No credit without explanation.*

Solution:

Call the teams A and B. In order for the series to end after four games, one team must win the first four games in a row. Thus the possible sequences are AAAA and BBBB (listing in an obvious notation the winner of each game). Any particular sequence of four games occurs with a probability $\left(\frac{1}{2}\right)^4$. Thus there is a $2/16 = 1/8$ chance the series ends in four games.

Next consider the chance that the series ends in five games. For team A to win the series in five games, it must win the last game, and three of the first four games. There are four possible sequences: BAAAA, ABAAA, AABAA, AAABA. To see this combinatorially, we know that A wins three of the first four games, so there are "four choose three" or $\binom{4}{3} = 4$ ways to choose positions for those wins.

Now reasoning as in the last paragraph, any particular sequence of five games occurs with a probability $\left(\frac{1}{2}\right)^5 = 1/32$. Thus there is a 4/32 chance of A winning in five games and by symmetry, a 4/32 chance of B winning in five games and overall an 8/32 = 1/4 chance that the series ends in five games.

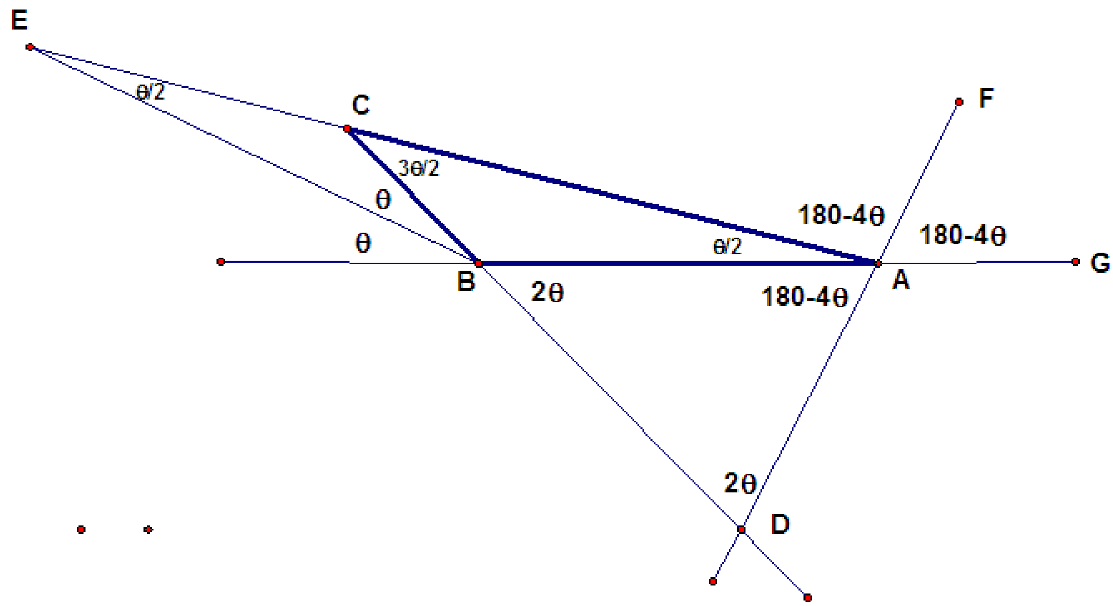
By similar reasoning, for team A to win the series in six games, it must win the last game and three of the first five. Therefore there are $\binom{5}{3} = 10$ possible sequences. Any particular sequence of six games occurs with a probability of $\left(\frac{1}{2}\right)^6 = 1/64$, so there is a $10/64$ chance that A wins in 6 games and hence a $20/64 = 5/16$ chance the series ends in six games. Now we could extend the counting argument to seven games, but since the series must be over by the seventh games, the total probability must add to 1. Thus we can subtract the sum of the preceeding probabilities from 1 to conclude that the chance that the series ends in seven games is $1 - 1/8 - 1/4 - 5/16 = 5/16$. Therefore we conclude that a six game series and a seven game series are equally likely.

Problem 5.

In triangle ABC, angle B is obtuse and $AB > BC$. The bisector of the exterior angle at A meets the extension of BC at a point D. The bisector of the exterior angle at B meets the extension of AC at a point E. If $AB = AD = BE$, find the measure of angle ADB.

Solution:

Using the figure below, triangle is ABC



Let $\angle CBE = \theta$, then $\angle ABD = 2\theta = \angle D$ since $AB = AD$

Then $\angle ABC = 180 - 2\theta$ and $\angle E = \angle EAB = \dfrac{\theta}{2}$ since $AB = BE$

This means $\angle ACD = \angle ACB = \dfrac{3\theta}{2}$

Finally, $\angle BAD = 180 - 4\theta = \angle FAG = \angle CAF$

Considering ΔABC, $\angle CAG = \angle ABC + \angle ACB$

Or $360 - 8\theta = \dfrac{3\theta}{2} + (180 - 2\theta)$, thus $\theta = 24$ and $\angle D = 2\theta = 48$

Problem 6.

Given $f(x) + f(y) = f(x+y) - xy - 1$ and $f(1) = 1$, find another integer N such that $f(N) = N$.

Solution:

Find $f(0)$ by letting $x = 1$ and $y = 0$; thus we get $1 + f(0) = 1 - 0 - 1$ and $f(0) = -1$.

Find $f(2)$ by letting $x = y = 1$; thus we get $1 + 1 = f(2) - 1 - 1$ and $f(2) = 4$

Find $f(3)$ by letting $x = 1$ and $y = 2$; thus we get $1 + 4 = f(3) - 2 - 1$ and $f(3) = 8$

So can see that as y gets large, $f(y)$ gets larger.

Find $f(-1)$ by letting $x = 1$ and $y = -1$; thus we get $1 + f(-1) = -1 + 1 - 1$ and $f(-1) = -2$

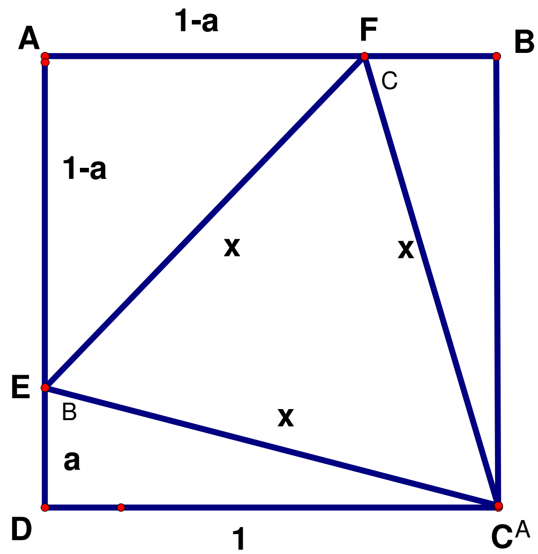
Find $f(-2)$ by letting $x = y = -1$; we get $-2 + (-2) = f(-2) - 1 - 1$ and $f(-2) = -2$

Therefore $N = -2$

Problem 7.
Find the area of the largest equilateral triangle that can fit within a unit square.

Solution:

In the figure below, in ΔAEF $x=(1-a)\sqrt{2}$ or $a=\dfrac{\sqrt{2}-x}{\sqrt{2}}$



In ΔCDE, $a^2+1=x^2$ so substituting for ***a*** we get $\left(\dfrac{\sqrt{2}-x}{\sqrt{2}}\right)^2+1=x^2$ or

$2-2x\sqrt{2}+x^2+2=2x^2$ or simplifying $x^2+2x\sqrt{2}-4=0$

Solving $x=\dfrac{-2\sqrt{2}\pm\sqrt{8+16}}{2}$ and $x=\sqrt{6}-\sqrt{2}$

Therefore area of $\Delta CEF=\dfrac{(\sqrt{6}-\sqrt{2})^2\sqrt{3}}{4}=\dfrac{(8-4\sqrt{3})}{4}\sqrt{3}=2\sqrt{3}-3$

Problem 8.

All numbers in this problem are written in base seven. Find the base seven ordered triple (x, y, z) satisfying the following system.

$$\begin{aligned} 11x - 5y - 12z &= 21 \quad (1) \\ 10x + 11y + 3z &= 15 \quad (2) \\ 22x + 15y \quad\quad &= 22 \quad (3) \end{aligned}$$

Solution:

Working in base seven, we have the following:

$$\begin{array}{lll} & 11x - 5y - 12z = 21 & \text{eq. (1)} \\ & 30x + 33y + 12z = 51 & \text{3 times eq.(2)} \\ \text{Add to get} & 41x + 25y \quad = 102 & \text{eq. (4)} \\ \text{Now} & & \\ \text{eq.(3)} \div 4 & 4x + 3y \quad = 4 & \text{eq. (5)} \\ & 153x + 111y \quad = 306 & \text{3 times eq. (4)} \\ & 136x + 111y \quad = 136 & 25_7 \text{ times eq. (5)} \\ \text{Subtract} & 14x \quad = 140 & \\ \text{Therefore} & x = 10 & \end{array}$$

Using eq. (3) $\quad y = \dfrac{22 - 220}{15} = \dfrac{-165}{15} = -11$

And using eq.(2) $\quad 10(10) + 11(-11) + 3z = 15$

$$3z = \frac{15 + 21}{3} = \frac{36}{3} = 12$$

Thus required base seven triple is $(x, y, z) = (10, -11, 12)$

Alternatively, you could convert the equations to base 10 to get:

$$\begin{aligned} 8x - 5y - 9z &= 15 \\ 7x + 8y + 3z &= 12 \\ 16x + 12y \quad &= 16 \end{aligned}$$

Solving these as above yields base 10 triple $(x, y, z) = (7, -8, 9)$ which equals $(10, -11, 12)_7$

Special Note:

1. **Students are asked to send their email address to barbara.unger735@gmail.com**
2. **The second test is available now at www.vtmathcoalition.org**
3. **The third test will be available on February 8, 2010 at www.vtmathcoalition.org**

The Math Coalition is grateful for problem contributors for this test including Middlebury College professors Michael Olinick, Bill Peterson, and Peter Schumer. Also contributing is Tony Trono, retired Burlington High School math teacher and Evan Dummit a graduate mathematics student at the California Institute of Technology.