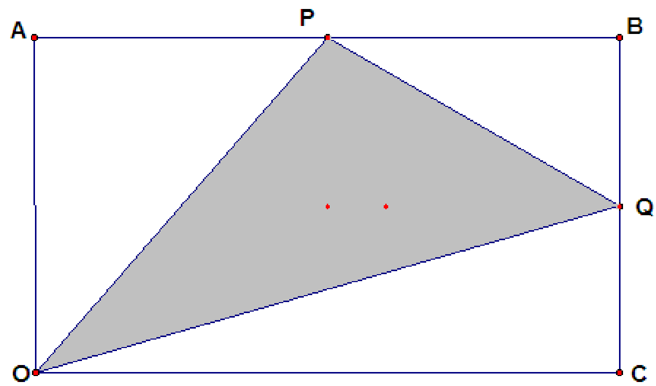
Vermont State Mathematics Coalition Talent Search November 2, 2009

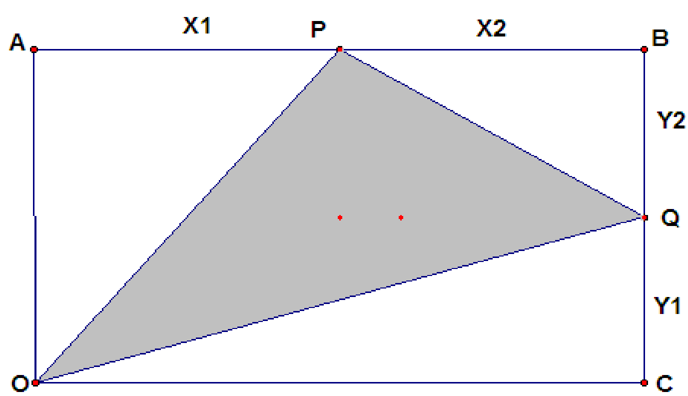
Test 2 Solutions

Problem 1.
Find the smallest positive integer which is evenly divisible by 225 and whose digits are all zeros or ones.
Solution:
The prime factorization of 225 is 3·3·5·5, so the number in question is a multiple of both 9 and 25. Recall that any multiple of 9 has digits which sum to 9. The smallest number divisible by 9 consisting of all 1's is therefore 111,111,111. Inserting any 0 digits only makes the result larger. Now observe that all multiples of 25 end in the digits 00, 25, 50 or 75, so our number must end with 00. It follows that the answer to our problem is 11,111,111,100.

Problem 2.
You are given a rectangle OABC from which you remove three right-angled triangles, leaving a fourth triangle OPQ as shaded in the diagram below.



How must you position the points P and Q so that the area of each of the three removed triangles is the same? That is, what are the ratios PB : PA and QB : QC?
Solution:



The diagram shows the rectangle with the lengths PA, PB, QB and QC labeled as x_1, x_2, y_1 and y_2 to simplify the algebra.

Equating the areas of ΔOAP, and ΔOCQ gives: $\frac{1}{2}x_1(y_1+y_2)=\frac{1}{2}(x_1+x_2)y_1$

Dividing both sides by $\frac{1}{2}x_1y_1$ gives $1+\frac{y_2}{y_1}=1+\frac{x_2}{x_1}$

Hence, it follows that ${y_2}/{y_1}={x_2}/{x_1}$. Denote this common ratio by ***r*.**

Now observe that this represents the unknown value of the ratios *PB:PA* and *QB:QC* that we need to find.

Now equating the areas of ΔOAP and ΔPBQ we have $\frac{1}{2}x_1\left(y_1+y_2\right)=\frac{1}{2}x_2y_2$

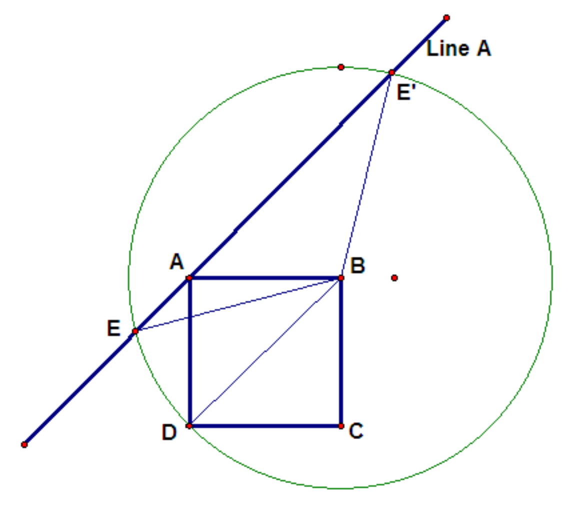
or $x_1y_1+x_1y_2=x_2y_2$. Dividing both sides of this equation by x_1x_2 yields $1+r=r^2$.

This is a quadratic equation with roots $\frac{1\pm\sqrt{5}}{2}$. For our problem, only the positive root is relevant, so we conclude that $r=\frac{1}{2}\left(1+\sqrt{5}\right)$ the golden ratio.

Problem 3.

Line A is drawn parallel to diagonal BD of square ABCD. A circle with radius BD and center B intersects line A at point E. Find angle ABE.

Solution:



Points E and E' can be interchanged so we must find angle E and E'.

Let $AB=1$, then $BD=BE=BE'=\sqrt{2}$.

$\angle ABD=\angle BAE'=45°$ and $\angle BAE=135°$

In $\Delta ABE'$, use Law of Sines:

$\frac{\sin E'}{1}=\frac{\sin 45}{\sqrt{2}}=\frac{1}{2}$ and $\angle E'=30°=\angle E$

$\angle BAE'+\angle E'=75°\quad\therefore\angle ABE'=105°$ and

$\angle BAE+\angle AEB=165°\quad\therefore\angle ABE=15°$

Two answers are $15°$ and $105°$

Problem 4.

The repeating decimal $x=0.\overline{8}$, $y=0.\overline{8}\overline{1}$ and $z=0.\overline{8}\overline{14}$ have a product which is a repeating decimal $0.\overline{ace}$. Find the sum of *a*,*c*,and *e*.

Solution:

$10x=8.888....$

$x=.8888....$ therefore $9x=8$ and $x={8}/{9}$

$100y=81.\overline{8}\overline{1}$

$y=.\overline{8}\overline{1}$ therefore $99y=81$ and $y={9}/{11}$

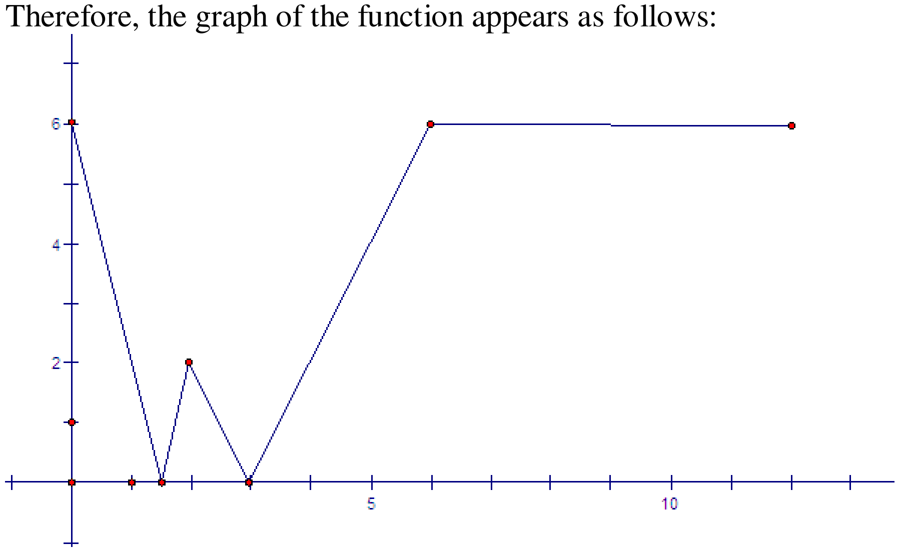
$1000z=814.\overline{8}\overline{14}$

$z = .\overline{814}$ therefore $999z = 814$ and $z = {}^{814}\!/_{999} = {}^{22}\!/_{27}$

Now product $xyz = \left(\frac{8}{9}\right)\left(\frac{9}{11}\right)\left(\frac{22}{27}\right) = \frac{16}{27} = .\overline{592}$ and sum $a + c + e = 16$

Problem 5.
Find the area of the region that lies under the graph of $f(x) = |||6 - x| - x| - x| - x|$ and above the x axis, between $x = 0$ and $x = 12$.
Solution:
First plot $f(x) = |||6 - x| - x| - x| - x|$. So suppose $x \geq 6$. Then
$f(x) = |||x - 6 - x| - x| - x| = |||-6| - x| - x| = ||6 - x| - x| = |x - 6 - x| = 6$

Now suppose $x \leq 6$, then $f(x) = |||6 - 2x| - x| - x|$
So for $6 \geq x \geq 3$ one has $f(x) = ||2x - 6 - x| - x| = ||x - 6| - x| = |6 - 2x| = 2x - 6$
While for $x \leq 3$ this becomes $f(x) = ||6 - 2x - x| - x| = ||6 - 3x| - x|$
so for $3 \geq x \geq 2$ this becomes $f(x) = |3x - 6 - x| = |2x - 6| = 6 - 2x$
and for $x \leq 2$ this becomes $f(x) = |6 - 3x - x| = |6 - 4x|$ which is equal to $f(x) = 4x - 6$ for $2 \geq x \geq 3/2$, and for $x \leq 3/2$, $f(x) = 6 - 4x$.
Therefore, the graph of the function appears as follows:



The area under f(x) consists of the following:
for $0 \leq x \leq 3/2$, a triangle of height 6 and base 3/2; therefore area is 9/2.

for $3/2 \leq x \leq 3$, a triangle with height 2 and base 3/2; therefore area is 3/2.

for $3 \leq x \leq 6$, a triangle with height 6 and base 3; therefore area is 9.

For $6 \leq 12$, a square with side equal 6; therefore area is 36.
Hence total area is 51.

Problem 6.
Alex is playing a card game and tabulating the results. He calculates that he has played 1800 games and won exactly 1542 of them. Rounded to the nearest percent, this is 86%.

What is the smallest number of consecutive games he would have to win in order for his winning percentage (rounded to the nearest percent) to be equal to 87%?

Solution:

Suppose he has played n games and won k of them, and wants his winning percentage to be at least γ after winning an additional x games in a row. Then $\frac{k+x}{n+x} > \gamma$ or $x > \frac{\gamma n - k}{1-\gamma}$.

Here, $\gamma = 0.865$, $n = 1800$, and $k = 1542$. Since $\gamma n = (0.865)(1800) = 1557$ we get $x > \frac{15}{0.135} = \frac{15000}{135} = 111.111...$ so the answer is 112.

Problem 7.

Find the smallest positive x which satisfies the inequality $x(x+1)(x+2)(x+3) \geq \frac{9}{16}$.

Solution:

Let $A = x(x+3) = x^2 + 3x$ and note that $(x+1)(x+2) = x^2 + 3x + 2$

Thus $A(A+2) \geq \frac{9}{16}$ and therefore $16A^2 + 32A - 9 \geq 0$.

Thus $(4A+9)(4A-1) \geq 0$; hence

$(4x^2 + 12x + 9)(4x^2 + 12x - 1) = (2x+3)^2(4x^2 + 12x - 1) \geq 0$

Since $x \geq 0$ solve $4x^2 + 12x - 1 = 0$ so $x = \frac{-12 \pm \sqrt{144+16}}{8} = \frac{-3+\sqrt{10}}{2}$

Problem 8.

Four consecutive numbers are $x, y, z,$ and w. The first three are in arithmetic progression and the last three are in geometric progression. If $x + w = 16$ and $y + z = 8$, evaluate $2009(x+y) + 50z + 95w$.

Solution:

Let the first three numbers be $a-2d, a-d, a$. Since $y, z,$ and w form a geometric progression, the ratio is $\frac{a}{a-d}$. Thus the four numbers are $a-2d, a-d, a, \frac{a^2}{a-d}$.

Since given that $x + w = 16$ we have $a - 2d + \frac{a^2}{a-d} = 16$ and given $y + z = 8$ we have $a - d + a = 8$ or $d = 2a - 8$. Substitute this in above equation:

$a - 2(2a-8) + \frac{a^2}{a-(2a-8)} = 16$ or $a - 4a + \frac{a^2}{8-a} = 0$ which simplifies to $a(a-6) = 0$.

Thus if $a = 6$, $d = 4$, then $(x, y, z, w) = (-2, 2, 6, 18)$ and the answer is 2010.

But if $a = 0$, $d = -8$, then $(x, y, z, w) = (16, 8, 0, 0)$ and an alternate answer is 48216.

The Math Coalition is grateful for problem contributors for this test including Middlebury College professors Michael Olinick, Bill Peterson and Peter Schumer. Also contributing is Tony Trono, retired Burlington High School math teacher and Evan Dummit a graduate mathematics student at the California Institute of Technology