Vertmont State Mathematics Coalition Talent Search March 31, 2010

Test 4 Solutions

Problem 1.

A very productive farmer finds that he has 7^{2009} eggs. He packs them into cartons each holding a dozen eggs, until he no longer has enough eggs to fill a carton. Then he takes the leftover eggs and makes an omelet. How many eggs go into his omelet?

Solution:

First, observe that $7^2 = 49 = (4 \cdot 12) + 1$. It follows that $7^{2008} = 49^{1004} = ((4 \cdot 12) + 1)^{1004}$. If we expanded the right hand side with the Binomial Theorem, then the last term would be 1, but every other term would have a factor of 12 to some positive power. It follows that $7^{2008} = 12n + 1$ for some integer n and $7^{2009} = 12m + 7$ for integer $m = 7n$. Thus the farmer will fill m cartons and have 7 eggs left over for his omelet. Equivalently, a more compact solution is available if students are familiar with modular arithmetic. Observe that $7^2 = 49 \equiv 1$ (mod 12). Thus $7^{2008} \equiv 1^{1004} \equiv 1$ (mod 12) and therefore $7^{2009} \equiv 7$ (mod 12). In other words 7^{2009} leaves a remainder of 7 when divided by 12.

Problem 2.

Make two random marks on a long stick.

(a) If you then break the stick into k pieces of equal length, what's the chance the two marks are on the same piece?

(b) If you then break the stick into k pieces of random lengths, what's the chance the two marks are on the same piece?

Solution: For the sake of argument, suppose the two marks are black and that the smaller pieces are created by first marking the break points with k - 1 yellow marks on the original long stick. The problem then is to find the probability that of the $k+1$ marks, the two black marks are consecutive.

(a) When all the pieces are of equal length, the k - 1 yellow marks are equally spaced. Once you make one black mark, the chance that the other is on the same piece is simply 1/k since it has an equal chance of being on any of the k equal-sized pieces.

(b) Interestingly, randomizing the breaking procedure nearly doubles your chance of the two marks being on the same piece.
There are "$k+1$ choose 2" = $(k+1)k/2$ ways to place the two black marks among the $k + 1$ total number of marks on the long stick. Of these, there are k ways that the two black marks are consecutive - 1st and 2nd, 2nd and 3rd, ..., kth and $(k+1)^{\text{st}}$. So the probability that they are consecutive (i.e. on the same piece) is k divided by $(k+1)k/2$. So the chance the two marks are on the same piece is $2/(k+1)$.

Problem 3.

. Find the sum of the solutions of $x^{1/4} = \dfrac{12}{7 - x^{1/4}}$

Solution:

Let $y = x^{1/4}$ (i.e the fourth root of x). So $x = y^4$. Then the original equation can be written as $y = \dfrac{12}{7-y}$. This simplifies to the quadratic equation $y^2 - 7y + 12 = 0$.

So $(y-3)(y-4) = 0$ and hence $y = 3$ or $y = 4$. Thus $x = 3^4 = 81$ or $x = 4^4 = 256$.

So the sum of the solutions is 81 + 256 = 337.

Problem 4.

Leonhard has ten rods having lengths 1, 2, ... ,10 respectively. How many different ways are there to make a triangle by choosing three appropriate rods?

Solution:

The key to the problem is simply to apply the triangle inequality (no side can be as long as the sum of the other two sides). Let the three sides of an allowable triangle be x, y, z with $1 <= x < y < z <= 10$. Now count all such x, y, z with $x + y > z$. In particular, let N(t) be the number of such triangles with x = t. Then N(1) = 0, N(2) = 7, N(3) = 11, N(4) = 12, N(5) = 10, N(6) = 6, N(7) = 3, N(8) = 1, and N(9) = N(10) = 0. So the total number is 50.

Problem 5.

A point is chosen randomly inside a square of side length 5, and a unit circle is drawn with that point as its center. Calculate the probability that the circle does not intersect either of the square's diagonals or any of its sides.

Solution:

If the squares side length is *n,* then the region in which the circle's center can lie is the set of points inside the square of side $n - 2$ with the same center as the original square (so that the circle does not go outside of the square) minus a region of points which are within 1 of either diagonal. A quick sketch indicates that this region is the union of four congruent isosceles right triangles. The length of the altitude to the hypotenuse of each triangle is easily seen to be $\dfrac{n}{2} - 1 - \sqrt{2}$, and the area of each triangle is $\left(\dfrac{n}{2} - 1 - \sqrt{2}\right)^2$.

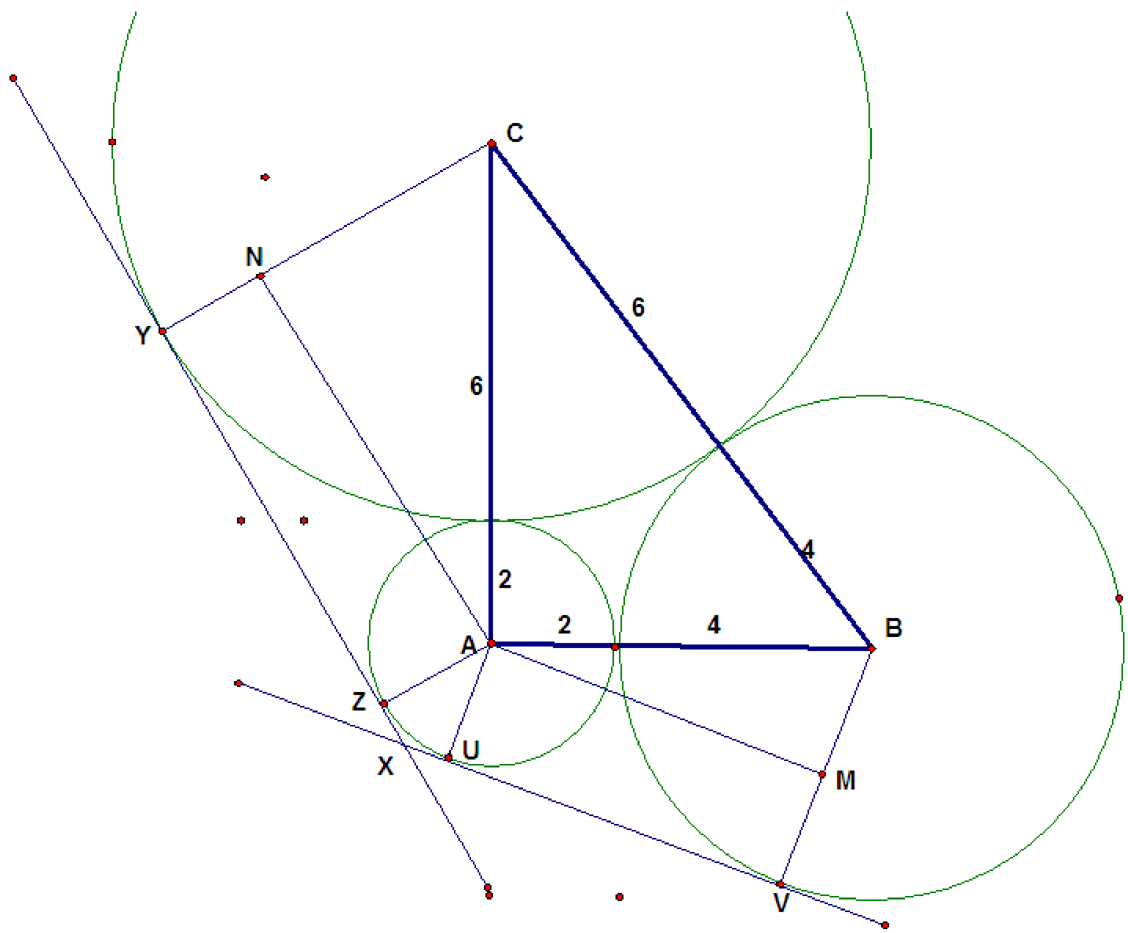
Since there are 4 triangles, the total area is $\left(n - 2 - 2\sqrt{2}\right)^2$. Since the total area in which the center can be placed is n^2, the probability is thus $\left(1 - \dfrac{2 + 2\sqrt{2}}{n}\right)^2$.

Setting $n = 5$ yields the desired probability $\left(\dfrac{3 - 2\sqrt{2}}{5}\right)^2 = \dfrac{17 - 12\sqrt{2}}{25}$

Problem 6.

Circles A, B and C with radii 2, 4, and 6 respectively are tangent to one another. The common external tangent to circles A and B intersects the common external tangent to circles A and C at point x. Find the measure of angle x.

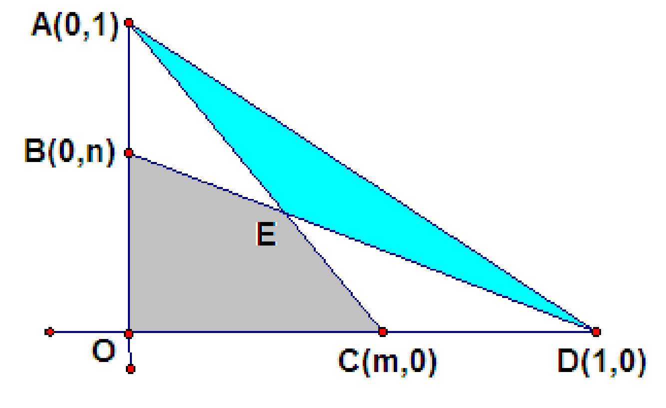
Solution:



Notice the three lines connecting the centers of the circles form a right triangle with sides 6, 8,and 10. Perpendiculars are drawn as shown with $AN \perp ZY$ and $AM \perp UV$. Thus angle $BAM = \arcsin\frac{1}{3}$ and angle $CAN = 30^\circ$. Let angle $UAZ = \phi$, then angles around A are :

$3(90)+30+\arcsin\frac{1}{3}+\phi = 360$ and $\phi = 60 - \arcsin\frac{1}{3}$ but angle at point x ($\angle UXZ$) and ϕ are supplementary; hence $\angle UXZ = 120 + \arcsin\frac{1}{3}$

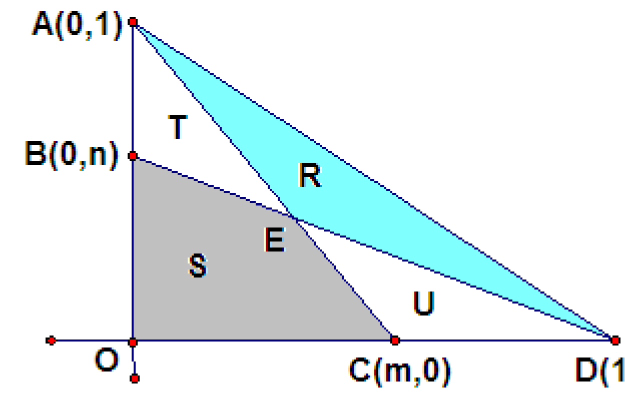
Problem 7.



Let the area of $\Delta AED = R$ and the area of $OCEB = S$. For $0 < m,\ n < 1$ express $|R - S|$ in terms of ***m*** and ***n*.**

Answer: ______________________

Solution:
Label the regions as shown.



Then

1. $R + T = area\Delta ABD = (1/2)(1-n)$
2. $R + U = area\Delta ACD = (1/2)(1-m)$
3. $R + S + T + U = area\Delta AOD = 1/2$

Since

$(R+T)+(R+U)-(R+S+T+U) = R-S$

thus $R - S = (1/2)(1-n+1-m-1)$ and

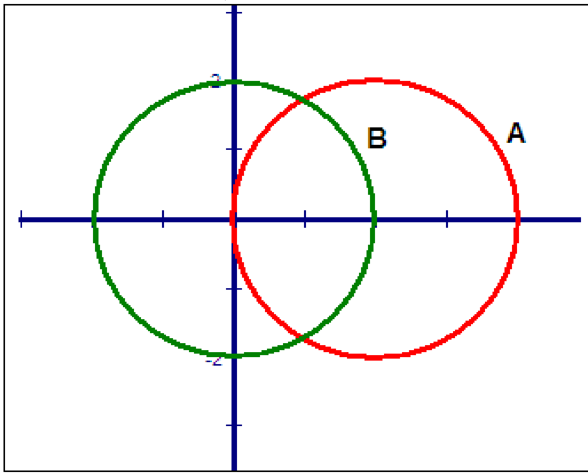
$|R-S| = \dfrac{1}{2}|1-m-n|$

Problem 8.

In the complex plan, let u and v be two distinct solution of $z^{2009}-1=0$. Find the probability that $|u+v|\geq 1$.

Solution:

Since the magnitude of all solutions = 1, and specifically $|u|=1$, we may equivalently seek the case in which $|1+\frac{v}{u}|\geq 1$, where $\frac{v}{u}$ is a randomly chosen root of $z^{2009}-1=0$ that is not equal to 1. A quick consideration of the unit circle shifted to the right by one, as shown below,



The circles intersect at ±2π/3. The x coordinate value of the red circle is less than the x coordinate value of the green circle between ±2π/3. Similarly the y coordinate value of the red circle is less (smaller) than the y coordinate value of the green circle between ±2π/3. The magnitude

$$|1+\frac{v}{u}|=\sqrt{\left(1+\cos(\frac{2\pi k}{2009}\right)^2+\sin^2\frac{2\pi k}{2009}}$$

indicates that the desired condition will hold precisely when the argument of $\frac{v}{u}$ is between $+\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$ inclusive, i.e. when the argument is $\frac{2\pi k}{2009}$ for $k\in\left[1,2,...,floor\frac{1}{3}2009, ceiling\frac{2}{3}2009,...,2008\right]$. Thus there are 669 + 669 = 1338 choices for $\frac{v}{u}$ which have the desired property, out of 2008 total choices. (root 1+i0 eliminated). Hence the probability is $\frac{1338}{2008}=\frac{669}{1004}$.

The Math Coalition is grateful for problem contributors for this test including Middlebury College professors Michael Olinick, Bill Peterson, and Peter Schumer. Also contributing is Tony Trono, retired Burlington High School math teacher and Evan Dummit a graduate mathematics student at the California Institute of Technology.