Vermont Talent Search **November 10, 2010**
School Year 2010-2011
Test 1 Solutions

Problem 1. Let Ş be the set of all ordered triples of rational numbers and let the operation Ω be defined as follows: $(a,b,c)\Omega(d,e,f)=(ad,bd+ce,cf)$. Find the inverse of $\left(4,-5,\frac{1}{6}\right)$ for the operation Ω.

Solution:
First, find the identity element (x,y,z) for the operation Ω.

$$(1,2,3)\Omega(x,y,z)=(1,2,3)$$

Thus, $1(x)=1$ or $x=1$

$3z=3$ or $z=1$

$2x+3y=2$ or $y=0$ and therefore $(x,y,z)=(1,0,1)$

Now let (u,v,w) be the inverse of $\left(4,-5,\frac{1}{6}\right)$. Then $\left(4,-5,\frac{1}{6}\right)\Omega(u,v,w)=(1,0,1)$.

Thus $4u=1$ or $u=\frac{1}{4}$ and $\frac{1}{6}w=1$ or $w=6$

$-5u+\frac{1}{6}v=0$ or $v=\frac{15}{2}$ and $(u,v,w)=\left(\frac{1}{4},\frac{15}{2},6\right)$

Problem 2.
In a sequence of 200 numbers, every one (except the end numbers) is equal to the sum of the two adjacent numbers in the sequence. The sum of all 200 numbers is equal to the sum of the first 100 numbers. The 32nd number is equal to 32. If the sequence is continued, find the sum of the 2010th and 2011th numbers.

Solution:
Let ***a*** be the first number and ***b*** the third number in the sequence, then the sequence can be written as: $a,a+b,b,-a,-a-b,-b,a,a+b,...$

Observe the sequence is periodic of length 6 and the sum of the 6 terms, $S_6=0$.

Thus $S_{100}=a+2b$ and $S_{200}=2a+b$ and since $S_{100}=S_{200}$ we conclude $a=b$.

The 32nd number is $a+b=32$ and hence $a=b=16$.

Thus the sequence is $16,32,16,-16,-32,-16,16...$

The 2010th number is -16 and the 2011th number is $+16$ so the sum is zero.

Note: Due to possible confusion on the test sheet, full credit will be given if the two numbers are correctly identified, but not summed.

The Vermont Math Coalition is grateful to problem contributors for this test including Tony Truno, retired Burlington High School Math teacher and Evan Dummit, a graduate mathematics student at the University of Wisconsin, Madison WI.

Problem 3.

The hypotenuse of each of four right triangles is equal to ***h***. The sum of the lengths of the shorter legs when taken three at a time is 89, 100, 104 and 127. The ratio of the areas of the two smaller triangles is $\frac{13}{33}$. Find the sum of the areas of the four triangles.

Solution:

Let the length of each triangles shortest leg be a,b,c,d.

Then
$$a+b+c=89$$
$$a+b+d=100$$
$$a+c+d=104$$
$$b+c+d=127$$

Adding the equations yields $3(a+b+c+d)=420$ or $a+b+c+d=140$

Subtract each of the four equations from $a+b+c+d=140$ and get $d=51,\ c=40,\ b=36,\ a=13$

Thus $\dfrac{\frac{13\sqrt{h^2-169}}{2}}{\frac{36\sqrt{h^2-1296}}{2}}=\dfrac{13}{33}$ Solving results is $h^2=7225$ and $h=85$

The four Pythagorean triples are $(13,84,85),(36,77,85),(40,75,85),(51,68,85)$.

The four areas are $(546,1386,1500,1734)$ respectively summing to 5166.

Problem 4.

In 3D space, the points $A=(a,0,0)$, $B=(0,b,0)$, $C=(0,0,c)$ lie on a XYZ coordinate axis where the origin is designated $O=(0,0,0)$. If the areas of $\Delta ABO=12,\ \Delta BCO=4$ and $\Delta ACO=6$ find the area of ΔABC.

Solution:

for ΔABO, area is $\frac{1}{2}ab=12$; $ab=24$

for ΔBCO, area is $\frac{1}{2}bc=4$; $bc=8$

for ΔACO, area is $\frac{1}{2}ac=6$; $ac=12$

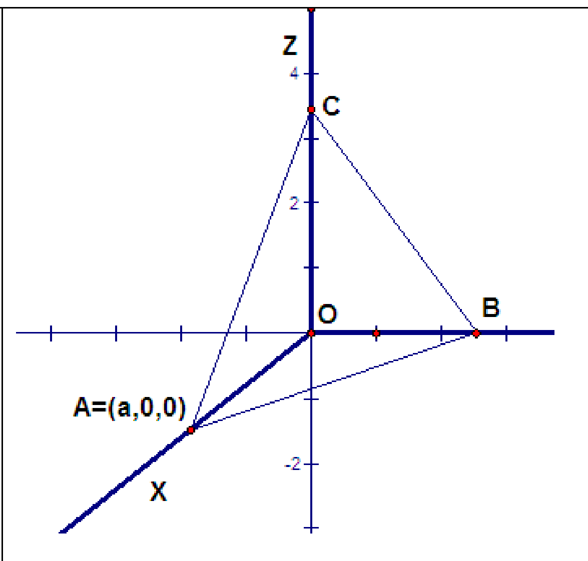
then multiplying

$a^2b^2c^2=(12)(8)(24)$

and $abc=48$

Since $ab=24,\ c=2$ and since

$bc=8,\ a=6$ and hence $b=4$



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Thus $AC = 2\sqrt{10},\ AB = 2\sqrt{13},\ \&\ BC = 2\sqrt{5}$.

Using Hero's formula: $S = \sqrt{5} + \sqrt{10} + \sqrt{13}$ and area of ΔABC is:

Area $\Delta ABC = \sqrt{\left(\sqrt{5}+\sqrt{10}+\sqrt{13}\right)\left(\sqrt{5}+\sqrt{10}-\sqrt{13}\right)\left(\sqrt{13}+\sqrt{5}-\sqrt{10}\right)\left(\sqrt{13}-\left(\sqrt{5}-\sqrt{10}\right)\right)}$

Now, note that the first two (and the last two) factors are differences of perfect squares, thus

Area $\Delta ABC = \sqrt{\left(15+2\sqrt{50}-13\right)\left(13-\left(15-2\sqrt{50}\right)\right)}$

Area $\Delta ABC = \sqrt{\left(2+10\sqrt{2}\right)\left(-2+10\sqrt{2}\right)} = \sqrt{-4+200} = 14$

Alternatively, compute $\cos A = \dfrac{(AC)^2 + (AB)^2 - (BC)^2}{2(AC)(AB)} = \dfrac{40+52-20}{2(2\sqrt{10})\left(2\sqrt{13}\right)} = \dfrac{9}{\sqrt{130}}$

Then $\sin A = \sqrt{1-\cos^2 A} = \dfrac{7}{\sqrt{130}}$ and

Area $\Delta ABC = \dfrac{1}{2}(AC)(AB)\sin A = \dfrac{1}{2}\left(2\sqrt{10}\right)\left(2\sqrt{13}\right)\dfrac{7}{\sqrt{130}} = 14$

Problem 5.

The following products are equal: $9(REDSOX) = 4(SOXRED)$.

Let T be the sum of the 6 digit numbers $REDSOX$ and $SOXRED$.

If $T+1 = b^n$, find the ordered pair (b,n).

Solution.

Let A and B be the 3 digit numbers RED and SOX respectively.

Then $9(1000A+B) = 4(1000B+A)$ or $8996A = 3991B$.

Dividing by 13 yields $692A = 307B$; since 692 and 307 have no common factor greater than 1 (relative primes) $A = 307$ and $B = 692$.

Thus $T+1 = 307692 + 692307 + 1 = 1000000$ and $(b,n) = (10,6)$.

Note: Equivalent formulations, for example $(1000,2)$*, will receive full credit.*

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Problem 6.
A set S is called 'special" if, for any two distinct elements x and y in S, the element $x+y$ is not in S. For example $\{1,3,5\}$ is special, but $\{1,3,4\}$ is not special. Find the largest integer n such that the set $\{2,3,4,...n\}$ may be written as the union of two special subsets.
Solution:
The decomposition $\{2,3,4,11,12\},\{5,6,7,8,9,10\}$ shows that 12 is possible. The following shows that 13 is *not* possible.
Assume 13 is possible and denote the two subsets as X and Y. We may assume that they are disjoint and also that 2 is in X. There are four cases to consider depending on the arrangement of 2, 3, and 4.
Case 1. Let 2, 3, and 4 be all in X. Then 5, 6, and 7 have to be in Y and we have $\{2,3,4\},\{5,6,7\}$. So 11 and 12 are in X and we have $\{2,3,4,11,12\},\{5,6,7\}$. Then 8, 9, and 10 must all be in Y yielding $\{2,3,4,11,12\},\{5,6,7,8,9,10\}$. But now 13 cannot be in either X or Y; a contradiction.
Case 2. 2 and 3 are in X and 4 is in Y. Then 5 is in Y and hence 9 cannot be in Y. So we have $\{2,3,9\},\{4,5\}$. Thus 6 and 7 cannot be in X so we get $\{2,3,9\},\{4,5,6,7\}$. But now 11 cannot be in either X or Y; a contradiction.
Case 3. 2 and 4 are in X and 3 is in Y. Then 6 cannot be in X and 9 cannot be in Y.So we have $\{2,4,9\},\{3,6\}$. But then 5 and 7 cannot be in X; hence 8 must be and we get $\{2,4,8,9\},\{3,5,6,7\}$. But now 12 can be in neither X or Y; a contradiction.
Case 4. 2 is in X and 3 and 4 are in Y. Then 7 is in X and 5 cannot be. So we have $\{2,7\},\{3,4,5\}$. But now 9 cannot be in either subset; a contradiction.
Since we have reached a contradiction in every case, our assumption is false and 13 is not possible. Indeed, our analysis in the first case shows that the only possibility with 12 is the one given above.

Problem 7
A 55 gallon fuel container is mistakenly filled with fuel containing 6% ethanol. How many gallons must be removed and then replaced with a 50% ethanol mixture in order that the resulting fuel solution is 10% ethanol.

Solution:
The ethanol removed plus the amount added must result in the required 10% concentration.
Let $x =$ gallons removed from the 55 gallon container.
Thus: $0.06(55-x)+0.5x=0.1(55)$

$$0.44x=2.2$$
$$x=5 \text{ gallons}$$

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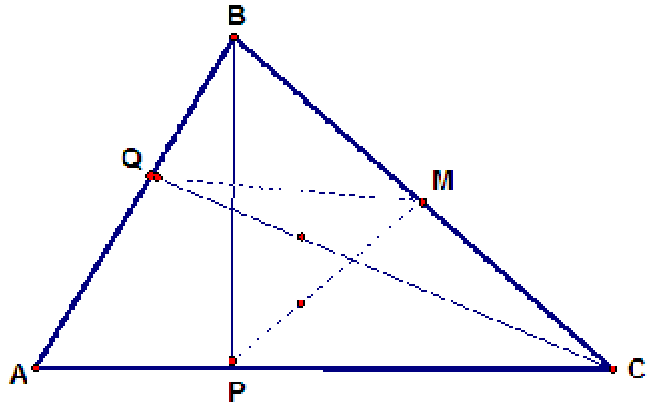
Problem 8.
In acute triangle ABC, altitudes BP and CQ are drawn with P on AC and Q on AB. If $CP = 2012$ and $BC = 2515$, find the sine of angle BQP.

Solution:

Note that $\frac{CP}{BC} = \frac{2012}{2515} = \frac{4(503)}{5(503)} = \frac{4}{5}$

Thus $BP = 3(503) = 1509$ and $\sin C = \frac{3}{5}$

Let M be the midpoint of BC. Thus in both right ΔBCP and ΔBCQ, the midpoint of the hypotenuse is equidistant from the vertices of the triangle.



This means $MB = MQ = MP = MC$ and quadrilateral $BCPQ$ is cyclic, that is, lie on a circle; note M is the center and MB, MQ, MP, and MC are radii.
Thus, angle C and angle BQP are opposite angles of an inscribed quadrilateral and are therefore supplementary and $\sin(\angle BQP) = \sin(\angle C) = \frac{3}{5}$.

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Test Two will be available at

http:/www.vtmathcoalition.org/talent-search/

on November 17, 2010

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