Vermont Talent Search **December 5, 2010**
School Year 2010-2011
Test 2 Solutions

Problem 1.
An ellipse E has two circles, $C1$ and $C2$, inscribed in it, such that $C1$'s center is the center of E and $C2$'s center is one of the two foci of E. If $C1$ is a unit circle and $C1$ and $C2$ are externally tangent, find the area of E.

Solution: Let A be the center of $C2$, X be a point of tangency of $C2$ to E, and B be the other focus of E. Also let £ be the tangent line to $C2$ and E at X. It is a well-known property of ellipses that AX and BX make the same (non-obtuse) angle with £ -- an equivalent interpretation of this statement is that a ray originating at one focus and bouncing off any point on the ellipse will pass through the other focus.

However, AX is perpendicular to £, since £ is the tangent line to the circle $C2$, and therefore BX is also perpendicular to £. So in fact A, B, and X are collinear, or, in other words, X lies on the major axis of E. Hence X is an endpoint of the major axis. Therefore, if the radius of $C2$ is r, the length of the major axis is $2 + 4r$. Now, since the sum of the distances of *any* point on an ellipse to the foci is constant, and the length of the semi-minor axis is 1, we can write $2\sqrt{1+(r+1)^2} = 2+4r$.

So $[1+(r+1)^2] = (1+2r)^2$ so $3r^2+2r-1=0$ and $r=\frac{1}{3}$ or $r=-1$, but since $r>0$ we conclude that $r=\frac{1}{3}$. Thus the semimajor axis of E has length $1+2r=\frac{5}{3}$ so the area of ellipse E is $\pi\left(\frac{5}{3}\right)(1)=\frac{5}{3}\pi$.

Problem 2.
Find the remainder when $f(x)=x^{57}+x^{19}+x^9+2010$ is divided by x^3-x.

Solution:
The remainder can be found by careful long division, but alternatively students may recall the division algorithm: If $f(x)$ and $g(x)$ are polynomials and the degree of $g(x)$ is less than or equal to the degree of $f(x)$, then there exists unique polynomials $q(x)$ and $r(x)$ such that $\frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}$; $g(x)\neq 0$ and the degree of $r(x)$ is less than the degree of $g(x)$. So let $g(x)=x^3-x=x(x+1)(x-1)$

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Then $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{ax^2+bx+c}{x^3-x}$ or $f(x) = x(x+1)(x-1)q(x) + ax^2+bx+c$

Thus

$f(0) = c = 2010$

$f(1) = 1+1+1+2010 = a+b+2010$ or $a+b=3$

$f(-1) = -3+2010 = a-b+2010$ or $a-b=-3$

Solving for a and b yields $a=0,\ b=3$, so remainder $r(x) = 3x+2010$

Problem 3.

In acute triangle ABC, altitudes AY and BX are drawn to sides BC and AC respectively. If $BY = 20$ and $AY = 99$, find the sine of angle AXY.

Solution:

Let AB be the diameter of a circle.

Thus the circle will pass through points X and Y as shown.

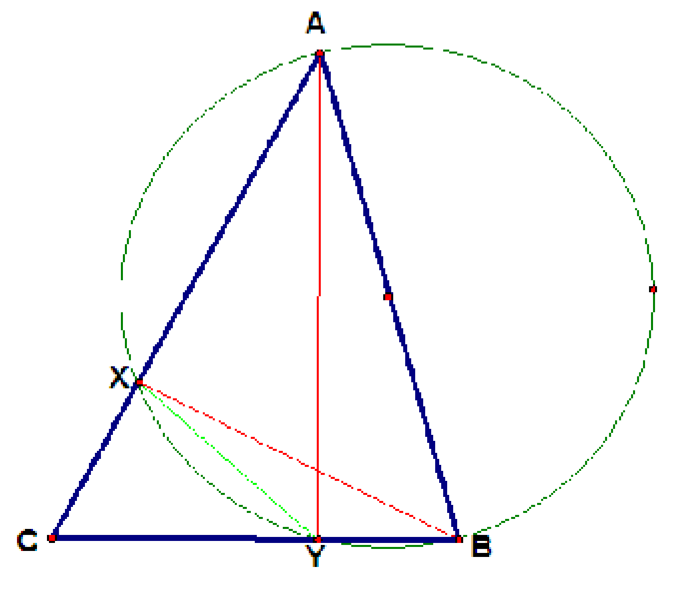
Since $(AB)^2 = 20^2 + 99^2$; $AB = 101$

Let $\angle BAY$ and $\angle BXY = \theta$ since they are both equal to $\{\frac{1}{2} arcBY\}$

$\angle AXY = 90 + \theta$

$\sin(90+\theta) = \cos\theta$

In ΔABY, $\cos\theta = \dfrac{99}{101}$



Problem 4.

The first, second and third terms of an arithmetic progression (AP) are x^2-6x+4, $3x^2-11x+2$, and $2x^2-x-12$ respectively.

The n^{th} term of the progression equals -2011. Find n.

Solution:

To find the difference in the arithmetic progression:

$(2x^2-x-12)-(3x^2-11x+2) = (3x^2-11x+2)-(x^2-6x+4)$

$3x^2-15x+12=0$ or $x^2-5x+4=0$

$(x-1)(x-4)=0$ and $x=1, 4$

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If $x = 4$, the A.P. is $-4, 6, 16...$ and no term is equal to -2011.

If $x = 1$, the A.P. is $-1, -6, -11...$ so use $l = a + (n-1)d$ to find n

$$-2011 = -1 + (n-1)(-5)$$

$402 = n - 1$ and finally $n = 403$

Problem 5.

It's well known that country music songs often emphasize love, prisons and trucks. A recent survey found the following:

a. 12 songs were about a truck driver who was in love while in prison.
b. 13 songs were about a prisoner in love.
c. 18 were about truck driver in love.
d. 28 were about a person in love.
e. 3 were about a truck driver in prison, but not in love.
f. 16 were about truck drivers who were not in prison.
g. 8 were about a person out of prison, who is not in love, and didn't drive a truck.
h. 2 songs were about a prisoner who was not in love and didn't drive a truck.

Find a) the number songs in the survey
b) the number of songs about
1. truck drivers
2. prisoners
3. truck drivers in prison
4. people not in prison
5. people not in love

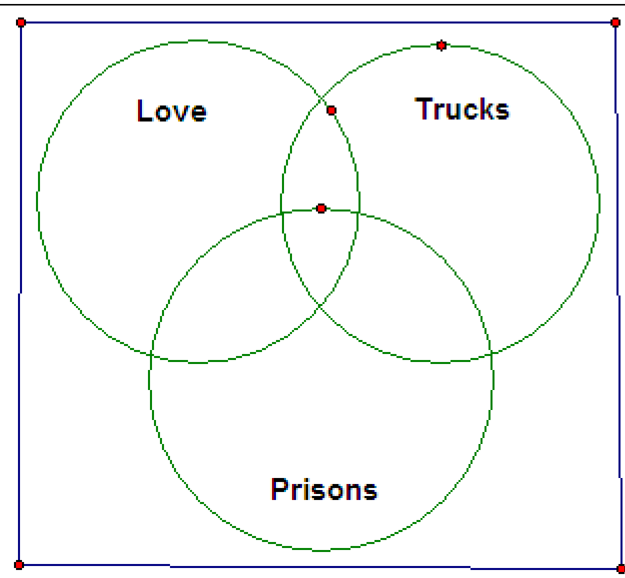
Solution:

It's helpful to drawn a Venn diagram as follows.

First immediately observe that there are 8 songs not included in love songs, trucks songs or prison songs as stated in the given information, Item *g*.
Next we see from Items *a*. and *e*. that there are $12 + 3 = 15$ truck drivers in prison.

Then we can use Item *b*. to conclude there is only 1 song about a prisoner in love.

Now we see, using Item *c*. that there must be 6 truck drivers in love but not in prison since $12 + 6 = 18$



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solution cont'd

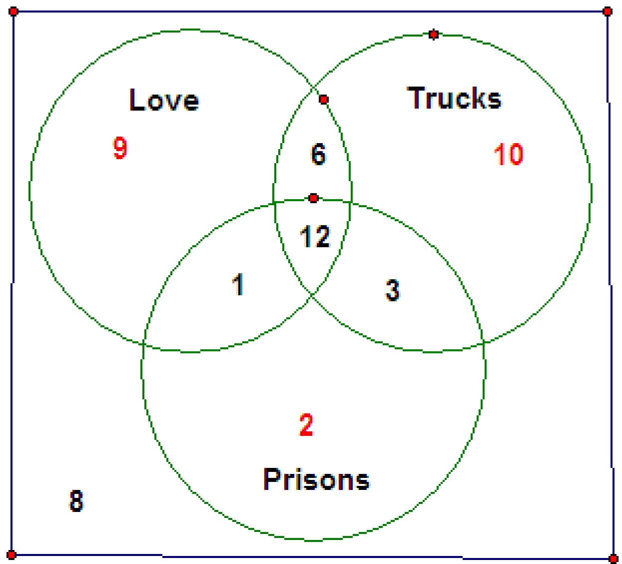
We now have this and using Item *h.* we can say there are $12+3+1+2=18$ prisoner songs.

Can also say, using Item *d.* there are $28-6-12-1=9$ songs about love only

Thus total number of songs surveyed is $10+6+12+3+2+1+9+8=51$

Finally since we determined above that 18 songs are about people in prison, there are $51-18=33$ songs about people not in prison.

Similarly there are $51-(9+6+12+1)=23$ about people not in love.



So in summary

a) the number songs in the survey — 51

b) the number of songs about

1. truck drivers	31
2. prisoners	18
3. truck drivers in prison	15
4. people not in prison	33
5. people not in love	23

Problem 6.

Three distinct integers are randomly chosen from the set $S=\{1,2,3...10\}$. What is the probability that their product is a perfect cube.

Solution:

We are looking for ordered triples (t,u,v) with $t<u<v$ and $t\cdot u\cdot v=n^3$. Observe that any prime dividing t must also divide n, since u and v are integers.

If $t=1$, then the maximum product is $1\cdot 10\cdot 9=90$ so the possibilities are 2^3, 3^3 and 4^3. For 2^3 we get $(1,2,4)$. For 3^3 only $(1,3,9)$ works and for 4^3 there is no solution.

If $t=2$ the maximum product is $2\cdot 10\cdot 9=180<216=6^3$ so the only possibility is 4^3 which yields $(2,4,8)$ since $4\cdot 8$ is the only way to get $\frac{4^3}{2}=32$.

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If $t=3$, the only possibility is 6^3, yielding $(3,8,9)$ since $8\cdot 9$ is the only possible way to get $\frac{6^3}{3}=72$.

If $t=4$, the maximum product is 360 so it is not possible to get 8^3, so 6^3 is the only possibility yielding $(4,6,9)$ since $6\cdot 9$ is the only way to get $\frac{6^3}{4}=54$.

If $t\geq 5$ there are no solutions since t^3 is not allowed. For $t\neq 8,9$ this follows immediately from the prime factors observation and for $t=8,9$ it is clear there are no more solutions.

So we have 5 ordered triples; namely $(1,2,4),(1,3,9),(2,4,8),(3,8,9)$ and $(4,6,9)$.

There are "10 choose 3" or $\binom{10}{3}=\frac{10\cdot 9\cdot 8}{6}=120$ possible ordered triples from set S so the answer is $\frac{5}{120}=\frac{1}{24}$

Alternate Solution:

Consider the prime factorization of each element in the set S.

1; **2**; **3**; **4** = 2x2; **5**; **6**=2x3; **7**; **8** =2x2x2; **9** = 3x3; **10**=2x5

Determine if the product of three of these elements will generate a perfect cube.

1^3 = 1 impossible; 2^3= 8 and 1 x 2 x 4 = 2x2x2 = 8; 3^3 = 27 and 1x3x9 = 3x3x3 = 27;
4^3 = 64 and 2x4x8 = 4x4x4 = 64; 5^3= 125 there are only two factors of 5 in the set S;
6^3 = 216 4x6x9 = 2x2x2x3x3x3 = 216 or 3x8x9 = 3x2x2x2x3x3 = 216;
7^3 impossible – there is only one factor of 7;
8^3 impossible – there are not enough factors of 2 or 8; 9^3 impossible – there are not enough factors of 3 or 9; 10^3 impossible – there are not enough factors of 5.

In the case 2^3 where 1 x 2 x 4 generates the product 8. There are 6 ways of choosing these 3 elements. 1x2x4; 1x4x2; 2x1x4; 2x4x1;4x1x2; 4x2x1. Considering the set S we have five possibilities for perfect cubes, we then have 6x5 or 30 chances to get a perfect cube.

The number of possible outcomes is 10 x 9 x 8 = 720. 30/720 = 1/24 as the probability of getting a perfect cube.

Problem 7

The equation $x^3+ax^2+bx+c=0$ has real non-zero roots. If two of the roots are addition inverses and two are multiplicative inverses, determine the maximum possible value of $(b+c)$.

Solution:

The roots must be as follows: $r,\ -r,$ and $\frac{1}{r}$.

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For the equation as given, the product of the roots is $c = -(r)(-r)\left(\frac{1}{r}\right) = r$

Sum of the root pairs: $b = -r^2 + 1 - 1 = -r^2$

Note: Many students may have committed to memory the relationship between polynomial roots and coefficient values, but these are also easily derived. Let $x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r)$.

Now expand and collect terms to get $a = -(p+q+r)$, $b = (pq + rp + rq)$, $c = -rpq$.

Thus $(b+c) = -r^2 + r$ which can be maximized by completing the square…

$$-\left(r^2 - r + \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} - \left(r - \frac{1}{2}\right)^2$$

As $r \to \frac{1}{2}$ the expression reaches a maximum value of $\frac{1}{4}$.

Problem 8.

A positive integer N leaves a remainder of 1 when divided by 3, a remainder of 3 when divided by 5, a remainder of 5 when divided by 7, a remainder of 7 when divided by 9 and a remainder of 10 when divided by 11. Find the smallest such N.

Solution:

First observe that subtracting 2 from any multiple of 3 leaves a remainder of 1 after division by 3.

Now smallest number, N, with a remainder of 3 when divided by 5 is 13 and 13 is indeed 2 less than a multiple of 3. To keep remainders the same after division by 3 and 5, add multiples of 15 to N until division by 7 yields a remainder of 5. This occurs at $N = 103$. Again, to keep constant remainders add multiples of $(3)(5)(7) = 105$ to 103 until division by 9 yields a remainder of 7. This results is $N = 313$. Now add multiples of $(5)(7)(9) = 315$ to 313 until sum is divisible by 11 with remainder of 10. Thus $N = 2518$.

Editor's Trivia Note:

In the Greek alphabet π is the 16th letter and 16 is the square of 4. In the English alphabet p is also the 16th letter and i is the 9th letter and 9 is the square of 3. Add them up (16+9), and you get 25, the square of 5.
Multiply them and you get 144, the square of 12.
Divide 9 by 16 and you get 0.5625 which is the square of 0.75.
Is it any wonder they say "π are squared"

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