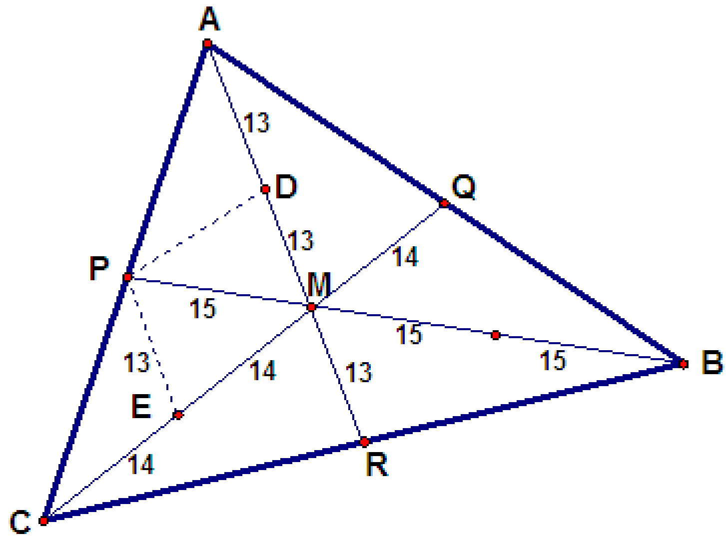
Vermont Talent Search **April 12, 2011**
School Year 2010-2011
Test 4 Solutions

Problem 1.
Find the area of a triangle whose medians have lengths of 39, 42 and 45.

Solution:

Let M be the center of gravity or centroid of the triangle.
Medians intersect at a point $\frac{2}{3}$ the distance from the vertices.
Let D bisect AM and E bisect CM. In ΔACM, PD and PE are midlines, i.e. join midpoints of sides, thus $PD = 14$ and $PE = 13$.
ΔDMP and ΔEMP are 13,14,15 triangles and their areas are 84.
(you can verify using Hero's formula)
Since D is a midpoint, the areas of ΔADP and ΔDMP are equal. Thus each of the 4 triangles in ΔACM has area 84. Similar results occur in ΔBCM and ΔABM. Therefore,



area of $\Delta ABC = 3 \cdot 4 \cdot 84 = 1008$.

Problem 2.
In quadrilateral $ABCD$, $\cot A = 4$, $\cot B = \frac{3}{2}$, $\cot C = 5$. Find all possible values for $\cot D$.

Solution:
Let $a = \tan A$, $b = \tan B$, $c = \tan C$ and $d = \tan D$

Then $a = \frac{1}{4}$, $b = \frac{2}{3}$ and so on.

We have $A + B + C + D = 2\pi$ and hence $\tan(A + B + C + D) = 0$.

Writing $A + B + C + D$ as $(A+B)+(C+D)$ and applying $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

yields: $\tan(A+B) + \tan(C+D) = 0$. Now applying the sum relation to both sums gives:

$\frac{a+b}{1-ab} + \frac{c+d}{1-cd} = 0$. Now cross multiply to get: $a + b - acd - bcd + c + d - abc - abd = 0$

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Solving for d gives $d = \frac{a+b+c-abc}{ab+ac+bc-1}$. Inserting values for a, b, c yields $d = -\frac{65}{39} = -\frac{5}{3}$ and hence $\cot D = -\frac{3}{5}$.

Problem 3.

In ΔABC, $AB = 4, BC = 5$ and $AC = 6$. Equilateral triangles ABD and CBF are drawn exterior to triangle ABC. CD and AF are drawn. Find the sum of $X + Y$ where $X = \angle ACD$ and $Y = \angle CAF$

Solution:

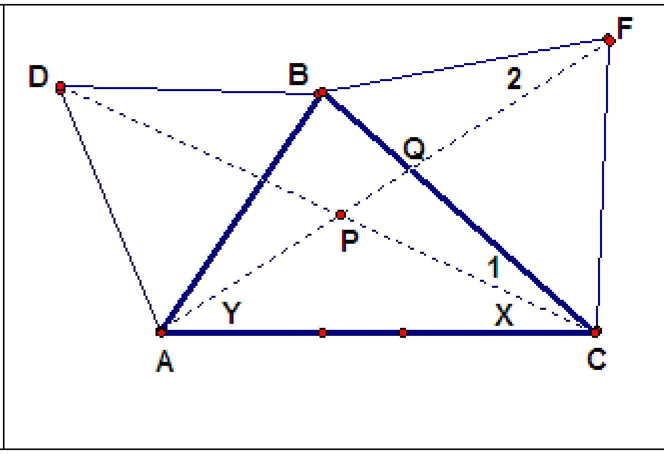
$\angle DBC = \angle ABF$; $\left(\angle B + 60^\circ\right)$

$BD = BA = 4$ and $BF = BC = 5$

$\Delta BCD \cong \Delta ABF$ by SAS and therefore $\angle 1 = \angle 2$ and since vertical angles are equal, $\Delta BFQ \approx \Delta PCQ$.

Thus $\angle FBQ = \angle CPQ = 60^\circ$

But $\angle CPQ$ is an exterior angle for ΔACP and $X + Y = 60^\circ$



Problem 4.

Given $f(n) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$.

If $f(n+1) - f(n-1) = kf(n)$ where k is an integer, find k.

Solution:

From the given equation we have

$$f(1) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)$$

$$f(-1) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{-1} + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{-1}$$

$$kf(0) = k\left(\frac{5+3\sqrt{5}}{10}\right) + k\left(\frac{5-3\sqrt{5}}{10}\right)$$

The equation is true for all n, so let $n = 0$, then $f(1) - f(-1) - kf(0) = 0$.

Therefore, factoring out terms we have..

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$$\left(\frac{5+3\sqrt{5}}{10}\right)\left[\frac{1+\sqrt{5}}{2}-\left(\frac{1+\sqrt{5}}{2}\right)^{-1}-k\right]+\left(\frac{5-3\sqrt{5}}{10}\right)\left[\frac{1-\sqrt{5}}{2}-\left(\frac{1-\sqrt{5}}{2}\right)^{-1}-k\right]=0$$

For the equation to be zero, the quantities in the brackets must be zero. Therefore

$\frac{1+\sqrt{5}}{2}-\frac{2}{1+\sqrt{5}}-k=0$ or $\left(6+2\sqrt{5}-4\right)-2k\left(1+\sqrt{5}\right)=0$

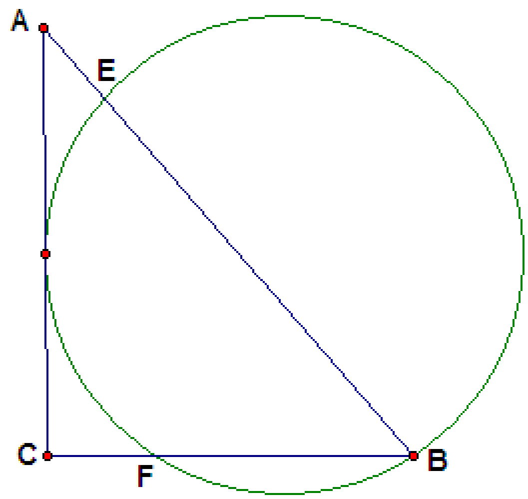
Thus $\left(2+2\sqrt{5}\right)\left(1-k\right)=0$ and $k=1$

Problem 5.

A circle is tangent to leg AC of right ΔABC and intersects the hypotenuse AB at E and leg BC at F. Point B is on the circles circumference.

AC and BC have integral lengths and $AC > BC$.

If $AE = 4$ and $BE = 21$ find the radius of the circle.

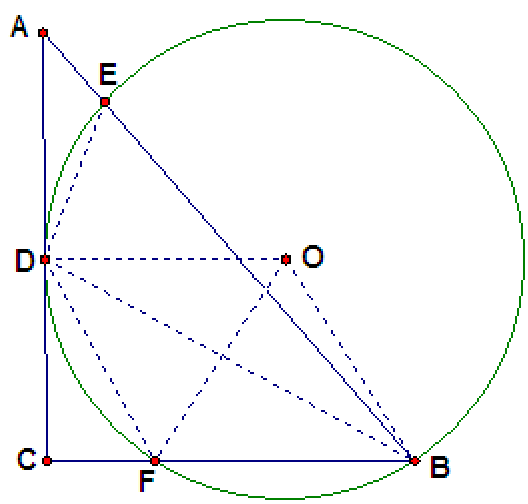


Solution:
Draw radii OD, OF, and OB. $OD \perp AC$.
Draw DF. Let $\angle DOF = \angle BFO = \theta$

Note that $\Delta AED \sim \Delta ADB$. Thus
$\left(AD\right)^2 = AE \cdot AB = 4 \cdot 25 = 100$ and $AD = 10$

The only right $\Delta's$ with hypotenuse of 25 are (7,24,25) and (15,20,25) and (7,24,25) is not possible. Thus AC=20, CD=10 & BC=15.
Note also that $\Delta DCF \sim \Delta DCB$; thus
$CF = \frac{20}{3}$. Hence

$$DF = \sqrt{100+\frac{400}{9}} = \frac{10\sqrt{13}}{3}$$



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Now use Law of Cosines in triangle DOF: $\left(DF^2\right)=r^2+r^2-2r^2\cos\theta$

and $\cos\theta=\dfrac{18r^2-1300}{18r^2}$. But in triangle BOF, which is also isosceles, $\cos\theta=\dfrac{25}{6r}$.

Equating these relationships and simplifying gives $18r^2-75r-1300=0$

$$(3r+20)(6r-65)=0$$

and $r=\dfrac{65}{6}$

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Problem 6.

Two numbers from the set $S = \{1, 2, 3, ..., 106\}$ are selected at random and multiplied. What is the probability that the product is a multiple of 5.

Solution:

Any number that is a multiple of 5 will produce a product that is a multiple of 5. There are 21 numbers is S that are multiples of 5 and 85 that are not multiples of 5.

Probability first number selected is multiple of 5 is $\frac{21}{106}$.

If there is no replacement, the second number is immaterial.

Probability first number not multiple of 5, but second one is equals $\frac{85}{106} \cdot \frac{21}{105}$

In this case, the probability product is a multiple of 5 equals: $\frac{21}{106} + \frac{85}{106} \cdot \frac{21}{105} = \frac{19}{53}$

With the first number replaced, the probability of either number being a multiple of 5 is $\frac{21}{106}$.

Equivalently, the probability of neither number being a multiple of 5 is $\left(\frac{85}{106}\right)^2$. Therefore the probability the product is a multiple of 5 is $\left(1 - \left(\frac{85}{106}\right)^2\right) = \frac{4011}{11236}$

Note: Due to the inexact problem statement, either answer or both receive full credit.

Problem 7

Let $a, b, c,$ and d be positive real numbers such that $\log_a b = c$, $\log_b c = 2d$, $\log_c d = 3a$, and $\log_d a = 4b$. Find the numerical value of the product $abcd$.

Solution:

From the given information we can say as follows:

$a^c = b,\ b^{2d} = c,\ c^{3a} = d,$ and $d^{4b} = a$

Thus the product

$$abcd = d^{4b} \cdot a^c \cdot b^{2d} \cdot c^{3a} = d^{4b} \cdot b \cdot b^{2d} \cdot d = b^{2d+1} \cdot d^{4b+1} = a^{c^{2d+1}} \cdot c^{(3a)^{4b+1}}$$

$$abcd = a^{(2cd+c)} \cdot c^{(12ab+3a)} = a^{2cd} \cdot a^c \cdot c^{12ab} \cdot c^{3a} = a^{2cd} \cdot b \cdot c^{12ab} \cdot d$$

$$ac = a^{2cd} \cdot c^{12ab} \text{ so } 2cd = 1 \text{ or } cd = \frac{1}{2} \text{ and } 12ab = 1 \text{ so } ab = \frac{1}{12} \quad abcd = \frac{1}{24}$$

Problem 8.

Find the sum $\sum_S \frac{1}{st}$ where S is the collection of all ordered pairs (s, t) of relatively prime positive integers such that $0 < s < t \le 12$ and $s + t > 12$.

Solution:

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The following solution will show by mathematical induction that $\sum_{S} \frac{1}{st} = \frac{1}{2}$ for any ***n*.**

Basic Step: Show that $\sum_{S} \frac{1}{st}$ is true for the base value $n = 2$. Clearly the only term in the sum is $(s,t) = (1,2)$ in which case the sum is $\frac{1}{2}$.

Inductive Step: Assume that $\sum_{S} \frac{1}{st} = \frac{1}{2}$ for $\nu = n$ and prove that $\sum_{S} \frac{1}{st} = \frac{1}{2}$ for $\nu = n+1$.
Specifically, the inductive step goes from adding all terms with $0 < s < t \le n$ and $s+t > n$ to instead adding all terms with $0 < s < t \le n+1$ and $s+t > n+1$. In doing so all terms with $s+t = n+1$ are removed and all terms with $t = n+1$ are added.
We claim the net effect of doing this is zero. To see this we need to show that
For $\nu < n+1-\nu$

$$\sum_{\substack{\gcd(\nu,n+1-\nu)=1 \\ \nu<n+1-\nu}} \frac{1}{\nu(n+1-\nu)} = \sum_{\gcd(s,n+1)} \frac{1}{(n+1)s} \qquad \text{EQ. 1}$$

where the LHS of EQ 1 is the sum of the removed terms and the RHS is the sum of the additional terms. Also, observe that ν and $n+1-\nu$ are relatively prime if and only if ν and $n+1$ are relatively prime. So we are summing over the same range.

Writing the summand on the LHS as $\frac{1}{\nu(n+1-\nu)} = \frac{1}{n+1}\left[\frac{1}{\nu} + \frac{1}{n+1-\nu}\right]$ and splitting into two sums gives the LHS as

$$\sum_{\substack{\gcd(\nu,n+1-\nu)=1 \\ \nu<n+1-\nu}} \frac{1}{\nu(n+1)} + \sum_{\substack{\gcd(\nu,n+1-\nu)=1 \\ \nu<n+1-\nu}} \frac{1}{(n+1)(n+1-\nu)}$$

Now interchange ν and $n+1-\nu$ in the second sum which gives just

$$\sum_{\substack{\gcd(\nu,n+1)=1 \\ \nu>n+1-\nu}} \frac{1}{(n+1)\nu}$$ which is the "missing half" of the sum on the other side.

So the net effect of going from n to $n+1$ is zero as claimed.

The following spreadsheet shows this in the case of $0 < s < t \le 12$ and $s+t > 12$.

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N=12

	S	T	S+T	$(ST)^{-1}$	
	1	12	13	0.083333	12
1	2	11	13	0.045455	22
2	3	11	14	0.030303	33
3	3	10	13	0.033333	30
4	4	11	15	0.022727	44
5	4	9	13	0.027778	36
6	5	12	17	0.016667	60
7	5	11	16	0.018182	55
8	5	9	14	0.022222	45
9	5	8	13	0.025	40
10	6	11	17	0.015152	66
11	6	7	13	0.02381	42
12	7	12	19	0.011905	84
13	7	11	18	0.012987	77
14	7	10	17	0.014286	70
15	7	9	16	0.015873	63
16	7	8	15	0.017857	56
17	8	11	19	0.011364	88
18	8	9	17	0.013889	72
19	9	11	20	0.010101	99
20	9	10	19	0.011111	90
21	10	11	21	0.009091	110
22	11	12	23	0.007576	132
23					
24			sum=	0.5	
25					
26					
27					

removed terms= 0.238709

N=13

S	T	S+T	$(ST)^{-1}$	
1	13	14	0.076923	
2	13	15	0.038462	26
3	13	16	0.025641	39
3	11	14	0.030303	33
4	13	17	0.019231	52
4	11	15	0.022727	44
5	13	18	0.015385	65
5	12	17	0.016667	60
5	11	16	0.018182	55
5	9	14	0.022222	45
6	13	19	0.012821	78
6	11	17	0.015152	66
7	13	20	0.010989	91
7	12	19	0.011905	84
7	11	18	0.012987	77
7	10	17	0.014286	70
7	9	16	0.015873	63
7	8	15	0.017857	56
8	13	21	0.009615	104
8	11	19	0.011364	88
8	9	17	0.013889	72
9	13	22	0.008547	117
9	11	20	0.010101	99
9	10	19	0.011111	90
10	13	23	0.007692	130
10	11	21	0.009091	110
11	13	24	0.006993	143
11	12	23	0.007576	132
12	13	25	0.00641	156
		sum=	0.5	

added terms= 0.238709

Note: Correct solutions supported with sound mathematical reasoning received full credit. It was not necessary to provide the solution by mathematical induction as shown above.

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