Test 1 Solutions

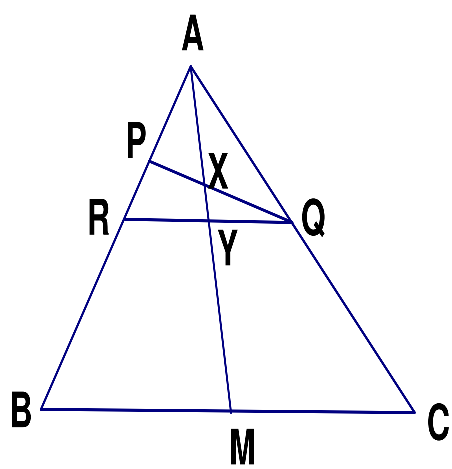
Problem 1

In ΔABC points P and Q are chosen on sides $\overline{AB}$ and $\overline{AC}$ respectively, and segment $\overline{PQ}$ is drawn meeting median $\overline{AM}$ at *X*. If $AP = \frac{1}{4}AB$ and $AQ = \frac{1}{2}AC$, find the ratio $\frac{PX}{PQ}$.

Solution:

Draw the line segment $\overline{RQ}$ parallel to $\overline{BC}$ as shown.

Let Y be the point where $\overline{RQ}$ meets $\overline{AM}$.
Since $\overline{YQ}$ is parallel to $\overline{MC}$ and passes through the midpoint Q of $\overline{AC}$, we know that $YQ = \frac{1}{2}MC$ and similarly $RQ = \frac{1}{2}BC$.
Since $MC = \frac{1}{2}BC$, it follows that $YQ = \frac{1}{2}RQ$ and Y is midpoint of $\overline{RQ}$. Since $\overline{RQ}$ is parallel to $\overline{BC}$ and Q is the midpoint of $\overline{AC}$ we conclude that R is the midpoint of $\overline{AB}$.
But $AP = \frac{1}{4}AB = \frac{1}{2}AR$ and P is the midpoint of $\overline{AR}$. Now we see that $\overline{AY}$ and $\overline{QP}$ are medians of ΔARQ and therefore their intersection point X lies 2/3 of the way from Q to P. Thus $\frac{PX}{PQ} = \frac{1}{3}$



Problem 2:

Ten points are chosen on a circle and all of the chords determined by these points are drawn.
(a) How many chords are there?
(b) Assume that no three of these chords intersect at a common point inside the circle. How many points inside the circle lie on two chords?

SOLUTION. Each of the chords has two of the original ten points as its endpoints, and every choice of two of these points determines a chord. The number of chords is therefore the number of ways of choosing two points from ten. This number is $\frac{10 \cdot 9}{2 \cdot 1} = 45$

Now each interior intersection point has exactly two chords going through it and thus it determines four distinct endpoints. Conversely, every choice of four of the original ten points determines six chords, but these have just one interior intersection point. The total number of interior intersection points is therefore equal to the number of ways of choosing four points from ten, namely

$$\binom{10}{4} = \frac{10!}{(4!)(6!)} = 210$$

Problem 3:

In trapezoid $ABCD$, AB parallel to CD, angle A is a right angle and $AB=4,\ AD=17,\ CD=12,$ and E lies on AD such that $\angle AEB = \frac{1}{2}\angle CED$. Find the ratio $AE:ED$.

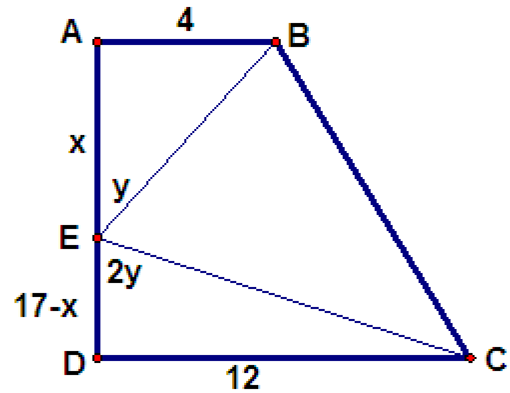
Solution:

Let $AE = x$, and then $ED = 17 - x$.

$\tan y = \frac{4}{x}$ and $\tan 2y = \frac{12}{17-x}$, but

$$\tan 2y = \frac{2\tan y}{1-\tan^2 y} = \frac{2\left(\frac{4}{x}\right)}{1-\left(\frac{4}{x}\right)^2} = \frac{8x}{x^2-16}$$

Thus $\frac{8x}{x^2-16} = \frac{12}{17-x}$



Cross multiply and simplify to get:

$5x^2 - 34x - 48 = 0$ or $(5x+6)(x-8) = 0$ and therefore

$x = 8,\ 17 - x = 9$ and $AE:ED = 8:9$

Problem 4:

Evaluate the sum $1\cdot(1+\frac{1}{n}) + 2\cdot(1+\frac{1}{n})^2 + 3\cdot(1+\frac{1}{n})^3 + \ldots + n\cdot(1+\frac{1}{n})^n$ in terms of n.

Solution:

This sum is of the form $S = k + 2k^2 + \ldots + nk^n$, where $k = 1 + \frac{1}{n}$. Multiplying both sides by k yields $kS = k^2 + 2k^3 + \ldots + nk^{n+1}$, so $S - kS = k + k^2 + \ldots + k^n - nk^{n+1}$.

But the sum $T = k + k^2 + \ldots + k^n$ is geometric and may be summed with the same trick namely $kT - T = k^{n+1} - k$ thus $T = \frac{k^{n+1}-k}{k-1}$ so

$S - kS = \frac{k^{n+1}-k}{k-1} - nk^{n+1} = (k-1)^{-1}\left[(n+1)k^{n+1} - k - nk^{n+2}\right]$ and finally

$S = (k-1)^{-2}\left[k + nk^{n+2} - (n+1)k^{n+1}\right]$ Now plug in $k = 1 + \frac{1}{n}$ and notice that

$nk^{n+2} - (n+1)k^{n+1} = 0$ to obtain simply $S = n^2 k = n^2 + n$

Problem 5:

If *a, b* and *c* are real numbers and $a+b+c=16,\ c^a = b^{2a},\ 2^c = 2\cdot 4^a$ and $abc<0$

Evaluate $9a-6b+9c$.

Solution:

From $c^a = b^{2a}$ we get $c=b^2$. From $2^c = 2\cdot 4^a$ we know that $2^c = 2^{2a+1}$ thus $c=2a+1$.

Combining results we get $a = \dfrac{b^2-1}{2}$ and therefore $\dfrac{b^2-1}{2} + b + b^2 = 16$ or $3b^2+2b-33=0$

Thus $(3b+11)(b-3)=0$ but when $b=3,\ c=9,\ a=4,\ abc>0$

Hence, $b=-\dfrac{11}{3},\ c=\dfrac{121}{9},\ a=\dfrac{56}{9}$ and $9a-6b+9c=199$

Problem 6:

If $a_n = 2 - \dfrac{n-1}{2}\log_{10} 5$ and $b_n = 8\cdot 100^{a_n}$, evaluate $\sum_{n=1}^{\infty} b_n$.

Solution:

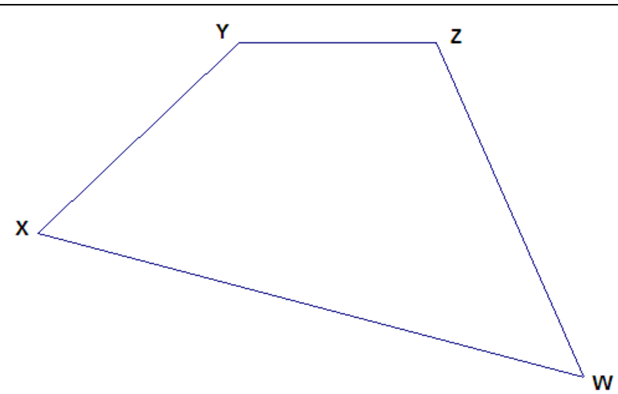
$$b_n = 8\cdot 100^{a_n} = 8\left(100^{2-\log 5^{\frac{n-1}{2}}}\right) = 8\left(\frac{100^2}{100^{\log 5^{\frac{n-1}{2}}}}\right) = 8\left(\frac{100^2}{10^{\log 5^{(n-1)}}}\right)$$

Thus $\sum_{n=1}^{\infty} b_n = 8\left(\dfrac{100^2}{5^{n-1}}\right) = 8\left(\dfrac{100^2}{1} + \dfrac{100^2}{5} + \dfrac{100^2}{25} + \ldots\right) = 8\cdot 100^2\left(\dfrac{1}{1-\dfrac{1}{5}}\right) = 100{,}000$

Problem 7:

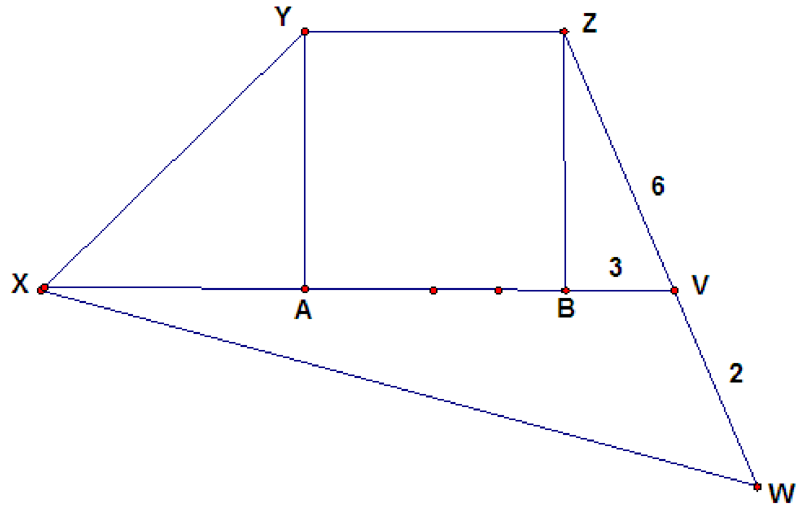
In quadrilateral $wxyz$, angle ***y*** is 135°, angle ***z*** is 120°, $xy = 3\sqrt{6},\ wz = 8,\ yz = 8-3\sqrt{3}$

Find the length of xw.



Solution:

Draw xv parallel to yz.
Draw yA and $zB \perp xv$
$\angle xyA = 45°$, $\angle vzB = 30°$
In ΔAxy, $Ax = Ay = 3\sqrt{3}$
In ΔBvz, $Bv = 3, vz = 6$
and $xv = 11$
Since $\angle xvw = 120°$
In Δxvw, use
Law of Cosines to
get
$(xw)^2 = 121 + 4 + 22) = 147$
$xw = 7\sqrt{3}$



Problem 8:
The *harmonic mean* of two positive real numbers *a* and *b* is the reciprocal of the arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$. A pair (a, b) of positive integers is called *dramatic* if the arithmetic mean of *a* and *b* equals 1 plus the harmonic mean of *a* and *b*.

If (a, b) is dramatic and $a \geq 2011$, find the minimal possible value for *b*.

Solution:
The problem statement says $\frac{2}{\frac{1}{a}+\frac{1}{b}} + 1 = \frac{a+b}{2}$ or $\frac{2ab}{a+b} = \frac{a+b-2}{2}$. Cross multiplying yields $4ab = a^2 + 2ab + b^2 - 2(a+b)$ or equivalently $2(a+b) = a^2 - 2ab + b^2 = (a-b)^2$
Since *a* and *b* are integers, we see that $a - b$ must be even; say $a - b = 2k$. Then we have $a + b = 2k^2$, so that (since $a - b = 2k$) $a = k^2 + k$ and $b = k^2 - k$.

Since $a \geq 2011$, we require $k \geq 45$ or $k \leq -46$, since $44^2 + 44 = 1980 = 45^2 - 45$, and $45^2 + 45 = 2070 = 46^2 - 46$. Thus the minimal a is 2070 and minimal *b* is 1980.