Vermont State Math Coalition **December 15, 2011**
Talent Search Test 2 Solutions

Problem 1:

Find the asymptotes of $f(x)=\dfrac{3x^2-5x-2}{x^2+5x-14}$

Solution:

Let $f(x)=\dfrac{3x^2-5x-2}{x^2+5x-14}=\dfrac{(3x+1)(x-2)}{(x+7)(x-2)}=\dfrac{3x+1}{x+7}$

Thus $f(x)$ tends toward infinity and x approaches -7.

Also, $f(x)=\dfrac{3x+1}{x-7}=\dfrac{(3+\frac{1}{x})x}{(1-\frac{7}{x})x}=\dfrac{(3+\frac{1}{x})}{1-\frac{7}{x}}\Rightarrow 3$ as $x\Rightarrow\infty$

Therefore asymptotes are $y=3$ and $x=-7$

Problem 2:

Find all real numbers x such that $\sqrt[3]{x+4}-\sqrt[3]{x}=1$

Solution:

Let $t=\sqrt[3]{x}$ and write the given equation as $\sqrt[3]{x+4}=1+t$

Now cube both sides to get $x+4=1+3t+3t^2+t^3$

Since $t^3=x$, this equation simplifies to $4=1+3t+3t^2$

Thus $t^2+t-1=0$ and the quadratic formula yields $t=\dfrac{-1\pm\sqrt{5}}{2}$

Since $x=t^3$, it follows there are only two solutions to problem statement namely ; $x=-2\pm\sqrt{5}$

Problem 3:

Suppose $a_0,a_1,a_2,\ldots,a_n$ are positive real numbers satisfying $a_ia_{n-i}=1$ for all $i=1,2,\ldots,n$.

If k is any integer, compute the sum $\dfrac{1}{1+a_0^k}+\dfrac{1}{1+a_1^k}+\dfrac{1}{1+a_2^k}+\ldots+\dfrac{1}{1+a_n^k}$

Solution:

Let

$$S=\frac{1}{1+a_0^k}+\frac{1}{1+a_1^k}+\frac{1}{1+a_2^k}+\ldots+\frac{1}{1+a_n^k}$$

Now write this sum in reverse order as follows:

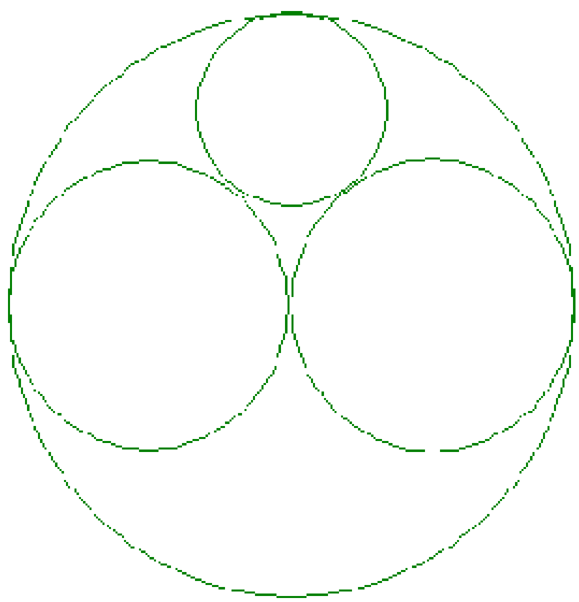
$$S = \frac{1}{1+a_n^k} + \frac{1}{1+a_{n-1}^k} + \frac{1}{1+a_{n-2}^k} + \ldots + \frac{1}{1+a_0^k}$$

Since $a_{n-i} = \frac{1}{a_i}$ we have

$$\frac{1}{1+a_i^k} + \frac{1}{1+a_{n-i}^k} = \frac{1}{1+a_i^k} + \frac{1}{1+a_i^{-k}} = \frac{1}{1+a_i^k} + \frac{a_i^k}{1+a_i^k} = 1$$

Now adding the two equations for **S** yields $2S = 1+1+1+\ldots+1 = n+1$ and $S = \frac{n+1}{2}$

Problem 4:



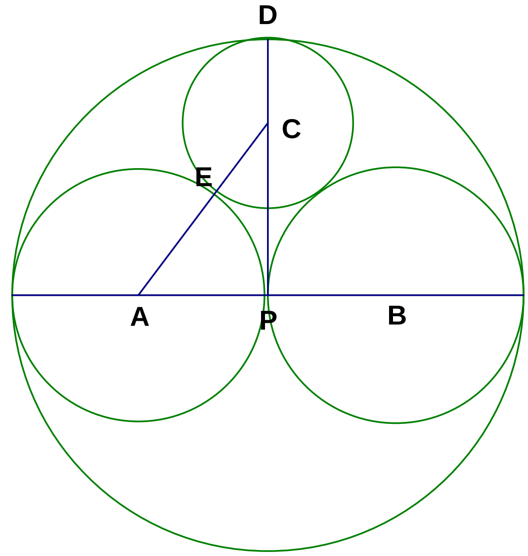
Shown here are four circles each tangent to the other three. The largest of these has radius 2 and each of the medium sized circles has radius 1.

Find the radius of the smallest circle

Answer: ___________________

Solution:

Let *A*, *B*, *C* and *P* be the centers of the four circles and observe that *P* is also the point of tangency of the two medium sized circles. Also, the common tangent line to these two circles must go through *C* and *D*, where the latter is the point of tangency of the smallest and largest circle. Furthermore, the line *AC* goes through point *E* where the circles centered at *A* and *C* are tangent.



Let *r* denote the radius of the small circle and consider the right triangle ΔCAP. We have $AP = 1$ and $AC = AE + EC = 1 + r$. Also $PC = PD - DC = 2 - r$ and the Pythagorean theorem yields $(1+r)^2 = 1 + (2-r)^2$.

Thus $1 + 2r + r^2 = 5 - 4r + r^2$ and $r = \frac{2}{3}$.

Problem 5:
Millie the cat wants to put on her socks and shoes. Her cabinet contains four identical socks and four identical shoes, and she draws them out one at a time in a random order. If she draws a sock, she puts it on one of her bare feet, and if she draws a shoe she will put it on a foot that has a sock, unless none of her feet have a sock, in which case she gives up. What is the probability that she is able to put on all of her socks and shoes without giving up?

Solution:
Label the socks *A* and the shoes *B*. We want to count the number of arrangements of *4A's* and *4B's* such that at each stage when we read the sequence from left to right there are at least as many *A's* as *B's*.
Clearly a sock must be drawn first, and a shoe must be drawn last. Let's break into cases based on the second and third objects; either two socks or a sock and a shoe.

i) a sock is drawn second and third. We have *AAA****B* and clearly the remaining *A* can go in any of the four remaining slots. Hence 4 cases.
ii) a sock is drawn second and a shoe third. We have *AAB****B*. The only issue in this case is if the fourth and fifth items drawn are both shoes; in all other

cases at least three of the first four objects are socks and the arrangement is valid. Thus there are $6-1=5$ ways to arrange the *A's* in this case.

iii) A shoe is second and a sock is third. We have *ABA****B*. If a sock is fourth, then the three remaining places for the last sock all work. If a shoe is fourth and we have *ABAB***B* then a third sock must be fifth and the fourth sock can be either sixth or seventh, for two more possibilities. Total 5 in this case.

Therefore a total of 14 arrangements will work.

Since there are eight chose 4 or $\binom{8}{4}=70$ sock/shoe arrangements total the requested probability is $\frac{14}{70}=\frac{1}{5}$

Problem 6:

Let $x_n=\dfrac{\log_2(n^{\sqrt{n}})\cdot\log_3(n^{\sqrt{n}})}{\log_3(n)+\log_{\sqrt{2}}(n)}$ for $2\le n\le 7$. Find the number of positive integers which divide $18^{x_2+x_3+x_4+x_5+x_6+x_7}$.

Solution:

Use logarithm rules to write $x_n=\dfrac{\log_2(n^{\sqrt{n}})\cdot\log_3(n^{\sqrt{n}})}{\log_3(n)+\log_{\sqrt{2}}(n)}$ as $x_n=\dfrac{\sqrt{n}\log_2(n)\cdot\sqrt{n}\log_3(n)}{\log_3(n)+2\log_2(n)}$.

Now divide top and bottom by $\log_2(n)\cdot\log_3(n)$ to get $x_n=\dfrac{\sqrt{n}\sqrt{n}}{\dfrac{1}{\log_2(n)}+\dfrac{2}{\log_3(n)}}$

Simplifying yields $x_n=\dfrac{n}{\log_n(2)+2\log_n(3)}=\dfrac{n}{\log_n(18)}=n\cdot\log_{18}(n)=\log_{18}(n^n)$.

Therefore, $18^{x_n}=n^n$ so the given integer is $2^2\cdot3^3\cdot4^4\cdot5^5\cdot6^6\cdot7^7=2^{16}\cdot3^9\cdot5^5\cdot7^7$.
The number of divisors is thus $(16+1)(9+1)(5+1)(7+1)=8160$.

Problem 7:

Evaluate the sum $\displaystyle\sum_{n=-2012}^{n=2012}\frac{1}{3^n+1}$

Solution:

Note the three middle terms of the sum are as follows

$\dfrac{1}{3^{-1}+1}+\dfrac{1}{3^0+1}+\dfrac{1}{3^1+1}=1+\dfrac{1}{2}$. Now add remaining terms $\dfrac{1}{3^{-n}+1}+\dfrac{1}{3^n+1}=\dfrac{3^n+1}{3^n+1}=1$

Thus there are 2012 terms of 1 plus the middle term of ½ and sum is 2012.5

Problem 8:
There is a unique rational number *p/q*, with $q < 1{,}000{,}000$, such that the decimal expansion of *p/q* begins with 0.01040916253649. Find the ordered pair (p, q).

Solution: $S = \frac{1}{1-a}\cdot T = \frac{a(1+a)}{(1-a)^3}$

Students should observe the given decimal expansion may be written as $\sum_{n=1}^{7}\frac{n^2}{100^2}$.

So we might guess that $\frac{p}{q}$ will be this sum taken to infinity; namely $\sum_{n=1}^{\infty}\frac{n^2}{100^n}$,
because this sum is a rational number and the terms past *n = 7* will be extremely small.

Now we wish to evaluate $S = \sum_{n=1}^{\infty} n^2 a^n$ for $a = \frac{1}{100}$. We can start be writing as follows:

$$a + 4a^2 + 9a^3 + 16a^4 + 25a^5 + \ldots =$$
$$= \left(a + a^2 + a^3 + a^4 + a^5 + \ldots\right) + 3\left(a^2 + a^3 + a^4 + a^5 + \ldots\right) + 5\left(a^3 + a^4 + a^5 + \ldots\right) + 7\left(a^4 + a^5 + \ldots\right) + \ldots$$

Each term inside the parentheses is a geometric series, which we can sum to obtain the new expression: $S = \frac{a}{1-a} + \frac{3a^2}{1-a} + \frac{5a^3}{1-a} + \frac{7a^4}{1-a} + \ldots = \frac{1}{1-a}\left(a + 3a^2 + 5a^3 + 7a^4 + \ldots\right)$
Now we must find the value of $T = a + 3a^2 + 5a^3 + 7a^4 + \ldots$ We can rewrite this series as $\left(a + a^2 + a^3 + a^4 + \ldots\right) + 2\left(a^2 + a^3 + a^4 + \ldots\right) + 2\left(a^3 + a^4 + \ldots\right) + 2\left(a^4 + \ldots\right) + \ldots$ and then sum the geometric series inside the parentheses again to obtain *T* as follows:

$$T = \frac{a}{1-a} + \frac{2a^2}{1-a} + \frac{2a^3}{1-a} + \frac{2a^4}{1-a} + \ldots = \frac{a}{1-a} + \frac{1}{1-a}\left(2a^2 + 2a^3 + 2a^4 + \ldots\right).$$

Here we can recognize yet another geometric series, so we have just
$T = \frac{a}{1-a} + \frac{1}{1-a}\cdot\frac{2a^2}{1-a} = \frac{a(1+a)}{(1-a)^2}$ and thus $S = \frac{1}{1-a}\cdot T = \frac{a(1+a)}{(1-a)^3}$. Hence for $a = \frac{1}{100}$
we obtain $S = \frac{100\cdot 101}{99^3} = \frac{10100}{970299}$. We guess this is the sought after value of *p/q* and indeed, to 25 decimal places this fraction equals 0.0104091625364964820122457 1.
Therefore the answer is (10100, 970299).