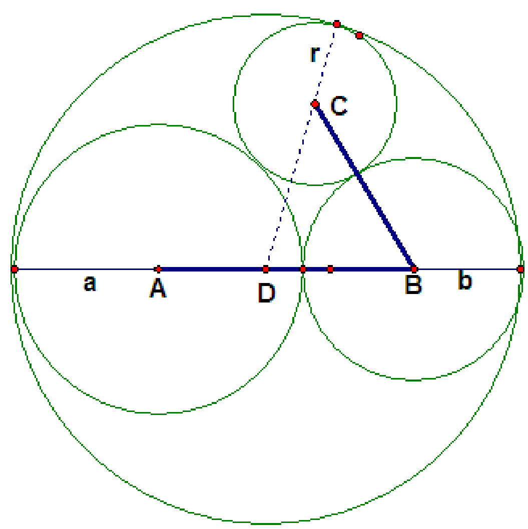
Vermont State Math Coalition **March 8, 2012**
Talent Search Test 3 Solutions

Problem 1:
Three circles are drawn inside another circle such that each of the four circles are tangent to the other three circles. Two circles with radii *a* and *b*, where $a > b$, have their centers on the diameter of the largest circle.

a) Find the radius of the largest circle in terms of *a* and *b*.
b) For what natural numbers *a* and *b* will the radius of the smallest circle be closest to one

Solution:



Let the radii of the three smallest circles be *a*, *b*, and *r*.
Therefore the radius of the largest circle is $a+b$ with center at *D*. Thus, $AD = b$ and $BD = a$

Now since in any two tangent circles, the two centers and the point of tangency are collinear we know that $CD = a+b-r$

Let $\theta = \angle ABC = \angle DBC$ and use Law of Cosines in ΔABC and ΔDBC

$$\Delta ABC: \ (a+r)^2 = (b+r)^2 + (a+b)^2 - 2(a+b)(b+r)\cos\theta$$

$$\Delta DBC: \ (a+b-r)^2 = a^2 - 2a(b+r)\cos\theta$$

Solving for $\cos\theta$ in both equations and equating values and solving for *r* gives

$$r = \frac{ab(a+b)}{a^2+ab+b^2}$$

Note sequence of values of $(a,1,r) = \frac{a^2+a}{a^2+a+1}$. Let $x = a^2+a$ and now we get $(a,1,r) = \frac{x}{x+1} = 1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots$ so as $x = a^2+a$ becomes larger $r \to 1$. Thus answer is $b=1$ and as $a \to \infty$, $r \to 1$.

Problem 2:
Let a,b,c and d be the distinct roots of $p(x) = x^4 + 2x^3 - 4x^2 - x + 1$. The six values $ab, ac, ad, bc, bd,$ and cd are roots of the polynomial *q(x)*. If $q(0)=1$, find $q(1)$.

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Solution:

Let $q(x)=e_6x^6+e_5x^5+e_4x^4+e_3x^3+e_2x^2+e_1x+e_0$

We have $p(x)=(x-a)(x-b)(x-c)(x-d)$. So by expanding and equating coefficients, we get $(a+b+c+d)=-2,\ (ab+ac+ad+bc+bd+cd)=-4$, $(abc+abd+acd+bcd)=1$ and $abcd=1$. For shorthand, we denote these "symmetric functions" as follows:

$$\sigma_1=a+b+c+d,\ \sigma_2=ab+ac+ad+bc+bd+cd$$
$$\sigma_3=abc+abd+acd+bcd \text{ and } \sigma_4=abcd$$

Now we can write $q(x)=e_6(x-ab)(x-ac)(x-ad)(x-bc)(x-bd)(x-cd)$ and $x=0$ yields $q(0)=1=e_6(-1)^6(abcd)^3$ so $e_6=1$. Furthermore, we can see that when we expand out the product for *q*(*x*), we obtain the coefficients of *q*(*x*) as some functions in *a, b, c, d* so to solve the problem, we must express the remaining coefficients e_5,e_4,e_3,e_2,e_1,e_0 of *q*(*x*) in terms of the symmetric functions $\sigma_1,\sigma_2,\sigma_3, and\ \sigma_4$.

We can group the terms of *q*(*x*) as follows:

$q(x)=[(x-ab)(x-cd)]\cdot[(x-ac)(x-bd)]\cdot[(x-ad)(x-bc)]$ yielding

$q(x)=[x^2-(ab+cd)x+\sigma_4]\cdot[x^2-(ac+bd)x+\sigma_4]\cdot[x^2-(ad+bc)x+\sigma_4]$

Expanding and equating coefficients then gives

$e_5=-(ab+cd+ac+bd+ad+bc)=-\sigma_2$

$e_4=3\sigma_4+[(ab+cd)(ac+bd)+(ab+cd)(ad+bc)+(ac+bd)(ad+bc)]$

$e_3=-2\sigma_4(ab+cd+ac+bd+ad+bc)-(ab+cd)(ac+bd)(ad+bc)$

$e_2=\sigma_4e_4$

$e_1=\sigma_4^2\cdot e_5$

$e_0=\sigma_4^3$

Now we need to observe that $(ab+cd)(ac+bd)(ad+bc)=\sigma_1^2\sigma_4+\sigma_3^2-4\sigma_2\sigma_4$ and $(ab+cd)(ac+bd)+(ab+cd)(ad+bc)+(ac+bd)(ad+bc)=\sigma_1\sigma_3-4\sigma_4$ which can be verified by multiplying out both sides of these equations. Thus we obtain:

$e_5=-\sigma_2=4$ $\quad$ $e_4=\sigma_1\sigma_3-\sigma_4=-3$

$e_3=2\sigma_2\sigma_4-\sigma_1^2\sigma_4-\sigma_3^2=-13$ $\quad$ $e_2=\sigma_4(\sigma_1\sigma_3-\sigma_4)=-3$

$e_1=\sigma_4^2(-\sigma_2)=4$ $\quad$ $e_0=\sigma_4^3=1$ and therefore

$q(x)=x^6+4x^5-3x^4-13x^3-3x^2+4x+1$ and $q(1)=-9$

Problem 3:

If $f(x)=\dfrac{3x+5}{3-5x}$, find all *x* for which $f(x^{-1})=f^{-1}(x)$.

Solution:

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$$f(x^{-1}) = \frac{\frac{3}{x}+5}{3-\frac{5}{x}} = \frac{3+5x}{3x-5}.$$

Now let $y = f^{-1}(x)$ and exchange *x* and *y* in *f(x)*, thus $x = \frac{3y+5}{3-5y}$ and solve for *y* getting $y = \frac{3x-5}{3+5x}$. Now equate $f(x^{-1})$ and $f^{-1}(x)$ and solve for *x*.

$\frac{3+5x}{3x-5} = \frac{3x-5}{3+5x}$ or $(3+5x)^2 - (3x-5)^2 = 0$ and $(8x-2)(2x+8) = 0$ thus $x = \frac{1}{4}, -4$

Problem 4:
Kathleen, Robert, Mary and Jordan are playing a game with a standard 6 sided die. They each roll the die, and the person who rolls the lowest number wins. If there is a tie, the people who tied each roll again and the lowest new roll wins. If there is still a tie, they continue rolling until someone wins the tiebreak. Jordan rolls first and gets a 3. What is the probability that he will win?

Solution:
First, if everyone else rolls a 4 or higher, Jordan will win without a tiebreak.
The probability of this is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
Clearly, if there is a tie, all of those participating in the tiebreak have an equal chance to win. So we can separate into cases depending on how many people tie with Jordan.
The probability that exactly one person ties with Jordan is $3\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{8}$, since there are 3 choices for the person to tie with him. Since Jordan has a 50% chance of winning the tiebreak, his probability of winning in this case is 1/16.
The probability that exactly two people tie with Jordan is $3\left(\frac{1}{6}\right)^2\left(\frac{1}{2}\right) = \frac{1}{24}$ since there are 3 choices for the person left out of the tiebreak. Since Jordan has a 1/3 chance of winning this tiebreak, his probability of winning in this case is $\frac{1}{72}$.
Finally, the probability that everyone rolls a 3 is $\left(\frac{1}{6}\right)^3$ and Jordan has only a 25% chance of winning this tiebreak, so his probability of winning this way is $\left(\frac{1}{6}\right)^3\left(\frac{1}{4}\right) = \frac{1}{864}$.
Therefore his overall probability of winning is $\frac{1}{8} + \frac{1}{16} + \frac{1}{72} + \frac{1}{864} = \frac{175}{864}$

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Problem 5

Three circles are drawn inside another circle such that each of the four circles are tangent to the other three circles. If the radii of the three inner circles are 1, 2, and 3, find the radius of the largest circle.

Solution

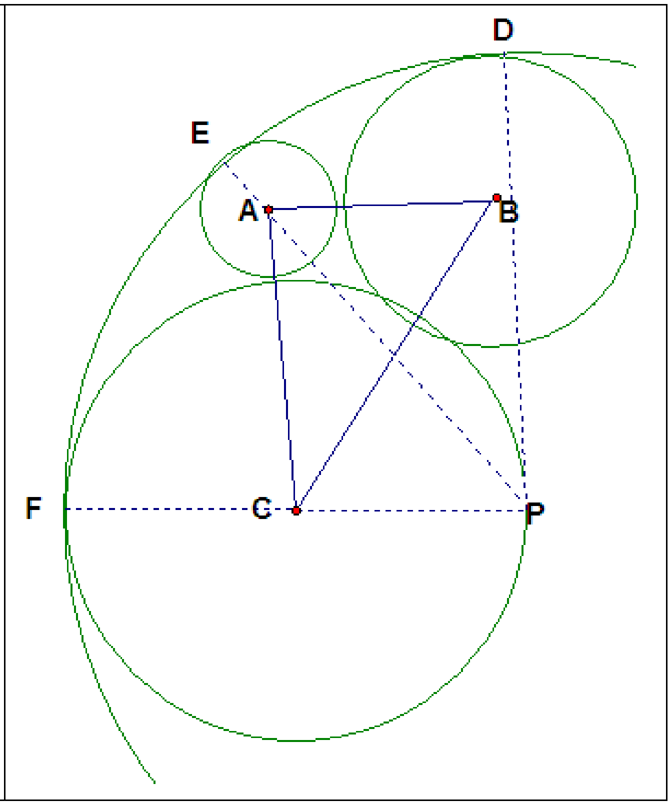
At A, B, and C draw circles with radii 1, 2, and 3 respectively.

Note that triangle *ABC* is a 3-4-5 right triangle.

Form rectangle *ABPC* by drawing BP parallel to AC and CP parallel to AB.

PD = 4+2 = 6 and D is the point of tangency. Also PF = 3+3 = 6 and F is a point of tangency and finally PE = 5+1 = 6 is a point of tangency.

Thus radius of largest circle is 6.



Problem 6

In algebra class, Yvonne was copying a graphing calculator problem from the board, but her teacher abruptly erased the problem before Yvonne finished writing. The portion of the problem that Yvonne was able to copy reads "Find the three values of *x* at which the graph of the line $y = 2x+1$ is tangent to the graph of $y = x^6 - 4x^5 + 2x^4 + 8x^3 -$". Given that Yvonne only missed the quadratic, linear and constant terms of the function, what is the answer to Yvonne's problem?

Solution:

Let Yvonne's polynomial be $p(x)$. If the line $y = 2x+1$ is tangent to the graph of the polynomial $y = p(x)$ at three places, then graph of the function $y = p(x) - (2x+1)$ is tangent to the *x*-axis in three places. The curve *y* being tangent to the *x*-axis requires it to have (at least) a double root there. But $p(x) - (2x+1)$ is a polynomial of degree 6 that has 3 double roots, hence it must be of the form $(x^3 + ax^2 + bx + c)^2$.

Therefore, we have,

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$$x^6 - 4x^5 + 2x^4 + 8x^3 - \ldots = \left(x^3 + ax^2 + bx + c\right)^2 = x^6 + (2a)x^5 + \left(a^2 + 2b\right)x^4 + (2ab + 2c)x^3 + \ldots$$

Equating coefficients gives $a = -2$, $b = -1$ and finally $c = 2$. Hence the points of tangency are the roots of the cubic

$(x^3 + ax^2 + bx + c) = x^3 - 2x^2 - x + 2 = (x+1)(x-1)(x-2)$ and answer to Yvonne's problem is $x = -1, 1, 2$. Note that we can also compute the polynomial

$$p(x) = \left(x^3 - 2x^2 - x + 2\right)^2 + 2x + 1 = x^6 - 4x^5 + 2x^4 + 8x^3 - 7x^2 - 2x + 5.$$

Problem 7

If $\sum_{n=1}^{99} \frac{1}{n(n+1)} = K$ and $KQ = 495$, find Q.

Solution:

Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

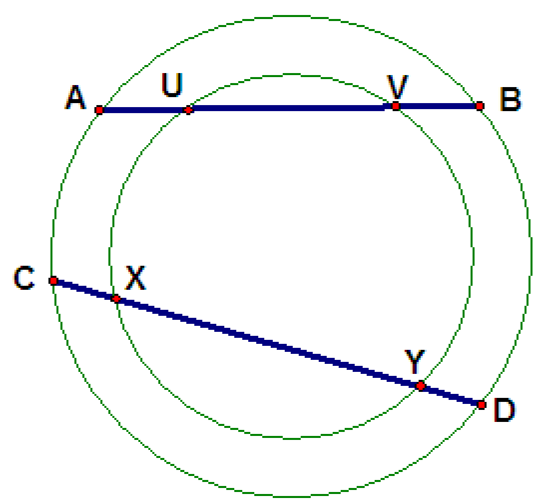
Thus $K = \sum_{n=1}^{99} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{98} - \frac{1}{99}\right) + \left(\frac{1}{99} - \frac{1}{100}\right)$

$$K = \sum_{n=1}^{99} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 - \frac{1}{100}\right) = .99 \text{ and therefore } Q = 500$$

Problem 8

Suppose chords *AB* and *CD* of a circle meet a smaller concentric circle at points *U*,*V*, *X* and *Y*.

If $AU = 2$, $UV = 10$ and $CX = 3$ find XY



Solution

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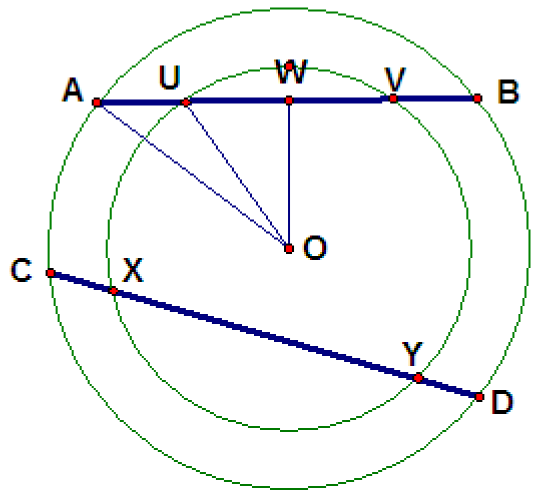
Let $r > s$ be the radii of the two circles and let O be the common center. Draw OW perpendicular to AB and draw lines AO and UO. Then we know that $AO = r$, $UO = s$ and W bisects chords AB and UV. If $UW = y$, then $UV = 2y$. Let $AU = x$ and $WO = z$ and we conclude from right triangles AWO and UWO that $r^2 = (x+y)^2 + z^2$ and $s^2 = y^2 + z^2$. Subtracting $r^2 - s^2 = x^2 + 2xy = x(x + 2y) = AU \cdot AV$ Similarly = $CX \cdot CY = r^2 - s^2 = AU \cdot AV$ and we conclude that $CY = 8$ and thus $XY = 8 - 3 = 5$.	

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