Test 4 Solutions

Problem 1:
Triangle ABC has $AB = 10$ and $12BC = 13AC$. Find the maximum area for ΔABC.

Solution:
Let $BC = 26x$ and $AC = 24x$.(See Note below) If K is the area of ΔABC, then Heron's formula gives
$K^2 = (5+25x)(-5+25x)(5-x)(5+x) = 5^2\left(25x^2-1\right)\left(25-x^2\right)$. Letting $y = x^2$, we obtain
$K^2 = 25(25y-1)(25-y) = 25\cdot\left(-25y^2+626y-25\right)$. This quadratic in y, and therefore the maximum occurs for $y = \frac{626}{2\cdot 25} = \frac{313}{25}$. Then
$K^2 = 25\cdot(25y-1)(25-y) = 25\cdot 312\cdot\frac{312}{25} = 312^2$. Hence the maximum area is 312 and it occurs when $BC = \frac{26\sqrt{313}}{5}$, $AC = \frac{24\sqrt{313}}{5}$

Note: If we let $BC = x$, then $AC = \frac{12}{13}x$, with fractions, the algebra becomes more tedious. Hence the choice of $BC = 26x$, which results in $AC = 24x$.

Problem 2:
Everyone has just one "magic birthday", when his age is exactly equal to the sum of the digits of the year of his birth. For example, the magic birthday of someone born in 1899 was in 1926. Notice that someone born in 1908 also had a magic birthday in 1926. Find the next year after 1926 in which two people born in different years can both have magic birthdays.

Solution.
The digit sum of every year before 1899 is at most 27, and so the magic birthday of a person born before 1899 was earlier than 1926. A person whose magic birthday occurs later than 1926, therefore, must be born in the year 1900 or later. For birth year $19ab = 1900+10a+b$ the digit sum is $10+a+b$ so the magic birthday occurs in year $1900+10a+b+10+a+b = 1910+11a+2b$

Similarly, for birth year 19cd, the magic birthday occurs in year $1910+11c+2d$
If these two magic birthdays occur in the same year, then

$$1910+11a+2b = 1910+11c+2d$$ and it follows that
$$11(a-c) = 2(d-b)$$

This forces (d - b) to be a multiple of 11. But b and d are digits so (d - b) lies somewhere between -9 and 9, so the only possibility is that $d-b=0$ and therefore $a-c=0$

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Thus $d = b$ and $a = c$ and for two people born in the years $19xx$ to have the same magic birthday in the same year, they must have been born in the same year.

However, the digit sum of 1982 is 20, and thus the magic birthday of someone born in 1982 will occur in 2002. Furthermore, a person born in the year 2000 will also have a magic birthday in 2002. Thus 2002 is one of those unusual years in which two people born in different years have a magic birthday.

Problem 3
Find all 3-tuples of complex numbers (x, y, z) satisfying the following system of equations:

$$(x^2 + xz)(y^2 + yz) = 36$$
$$(x^2 + xy)(z^2 + yz) = 225$$
$$(y^2 + xy)(z^2 + xz) = 100$$

Solution:
Rewrite the three equations as follows:

$$xy(x+z)(y+z) = 36$$
$$xz(x+y)(y+z) = 225$$
$$yz(x+y)(x+z) = 100$$

Multiplying them together and taking the square root yields

$$xyz(x+y)(x+z)(y+z) = \pm 900$$

And now dividing each in turn into the above yields

$$xz + yz = \pm 25$$
$$xy + yz = \pm 4$$

$xy + xz = \pm 9$ where the three choices of sign coincide.

Adding these three equations and dividing by 2 gives
$xy + xz + yz = \pm 19$ and then subtracting each in turn from this one gives

$$xy = \pm 6$$
$$xz = \mp 15$$
$$yz = \mp 10$$

Multiplying these three equations and taking the square root yields
$xyz = \pm 30,\ \pm 30i$ and finally dividing the equations into this one gives the four solutions:
$(x, y, z) = (3i, 2i, -5i),\ (-3i, -2i, 5i),\ (3, 2, -5),\ (-3, -2, 5)$

Note: Although most students who answered this problem provided all 4 3-tuples. I have decided to extend full credit to any student who answered with only the first two of the above 3-tuples. Students should note that the real numbers are typically considered a subset of the complex numbers and all 4 3-tuples were expected in the answer.

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Problem 4

The equation $2\cdot\left[\sqrt{x}-\cfrac{1}{\sqrt{x}-\cfrac{1}{\sqrt{x}-\cfrac{1}{\sqrt{x}-\dots}}}\right]^2 = x+\cfrac{1}{x+\cfrac{1}{x+\dots}}$ has a unique real positive solution for *x*. Given that $x \cong 4.969$, find *x* exactly in the form $\frac{a\sqrt{b}+c}{d}$ where *a, b, c, d* are integers.

Solution:

Let $y=\sqrt{x}-\cfrac{1}{\sqrt{x}-\cfrac{1}{\sqrt{x}-\dots}}$. Then $y=\sqrt{x}-\frac{1}{y}$ so $y^2-\sqrt{x}\,y+1=0$. Solving the quadratic equation yields $y=\frac{\sqrt{x}\pm\sqrt{x-4}}{2}$. Similarly if we let $z=x+\cfrac{1}{x+\cfrac{1}{x+\dots}}$, then $z^2-xz-1=0$ and $z=\frac{x\pm\sqrt{x^2+4}}{2}$. Since we clearly must have $z>0$, we know the plus sign is correct.

Thus we have the equation

$$2\cdot\left[\frac{\sqrt{x}\pm\sqrt{x-4}}{2}\right]^2=\frac{x+\sqrt{x^2+4}}{2} \text{ or } \left[\sqrt{x}\pm\sqrt{x-4}\right]^2 = x+\sqrt{x^2+4}$$

Now let $\alpha = x+\sqrt{x^2+4}$. **(1)**

We may multiply both sides by $\left(x-\sqrt{x^2+4}\right)$ obtaining $\alpha\left(x-\sqrt{x^2+4}\right)=x^2-(x^2+4)$ or

$$\frac{4}{\alpha}=-x+\sqrt{x^2+4} \qquad \textbf{(2)}$$

Now subtracting **(2)** from **(1)** yields $\alpha-\frac{4}{\alpha}=x+\sqrt{x^2+4}-(-x+\sqrt{x^2+4})=2x$ and thus $x=\frac{\alpha}{2}-\frac{2}{\alpha}$. For $\alpha=\left[\sqrt{x}\pm\sqrt{x-4}\right]^2=\left(2x-4\right)\pm2\sqrt{x^2-4x}$ we have

$$\frac{1}{\alpha}=\frac{1}{\left[\sqrt{x}\pm\sqrt{x-4}\right]^2}\cdot\frac{\left[\sqrt{x}\mp\sqrt{x-4}\right]^2}{\left[\sqrt{x}\mp\sqrt{x-4}\right]^2}=\frac{\left[\sqrt{x}\mp\sqrt{x-4}\right]^2}{4^2}=\frac{\left(2x-4\right)\mp2\sqrt{x^2-4x}}{16}$$

Hence we obtain $x=\frac{\alpha}{2}-\frac{2}{\alpha}=\frac{\left(2x-4\right)\pm2\sqrt{x^2-4x}}{2}-\frac{\left(2x-4\right)\mp2\sqrt{x^2-4x}}{8}$

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Multiplying both sides by 4 yields $4x = 3(x-2) \pm 5\sqrt{x^2-4x}$ or $x+6 = \pm 5\sqrt{x^2-4x}$
Since we know that *x* is positive, the plus sign must be correct. Squaring both sides yields $x^2+12x+36 = 25(x^2-4x)$. Since we took the plus sign on the square root, *x* is the larger root of the quadratic $24x^2-112x-36=0$ or $x = \frac{14+5\sqrt{10}}{6}$.

Problem 5:
An ordered 3-tuple (a,b,c) of positive integers is called a *cubic triple* if $\sqrt[3]{a+b\sqrt{c}} + \sqrt[3]{a-b\sqrt{c}} = 1$. For example, $(2,1,5)$ is a cubic triple because $\sqrt[3]{2+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$ and $\sqrt[3]{2-\sqrt{5}} = \frac{1-\sqrt{5}}{2}$ and their sum is obviously one. There exists a unique family of cubic triples given (given in terms of the parameter *t*) as $(a,b,c) = (\alpha_1 t+2, \alpha_2 t+1, \alpha_3 t+5)$ where α_1, α_2 and α_3 are pairwise coprime positive integers. Find the sum $\alpha_1+\alpha_2+\alpha_3$.

Solution:
Start by cubing both sides of $\sqrt[3]{a+b\sqrt{c}} + \sqrt[3]{a-b\sqrt{c}} = 1$ which yields the following. $\left(a+b\sqrt{c}\right) + 3\left(\sqrt[3]{a+b\sqrt{c}} + \sqrt[3]{a-b\sqrt{c}}\right)\cdot\left(\sqrt[3]{a^2-b^2c}\right) + \left(a-b\sqrt{c}\right) = 1$. Simplifying this equation yields $2a + 3\cdot\sqrt[3]{a^2-b^2c} = 1$ or equivalently $3\cdot\sqrt[3]{a^2-b^2c} = 1-2a$.

Now cubing both sides of this equation yields $27(a^2-b^2c) = (1-2a)^3$. Since (a,b,c) are integers, and the left-hand side is divisible by 3, we must have $(1-2a)$ divisible by 3 as well. Hence $a \equiv 2 \bmod 3$, so $a = 3k+2$ for some integer *k*. Substituting now into the equation above yields $27\left[(3k+2)^2 - b^2c\right] = (-3-6k)^3$. Solving for b^2c yields $b^2c = (1+2k)^3 + (3k+2)^2 = (k+1)^2(8k+5)$. Hence we can take $(b,c) = (k+1,\ 8k+5)$ and obtain $(3k+2, k+1, 8k+5)$ as a cubic triple for any positive integer *k*. Thus the sum $\alpha_1+\alpha_2+\alpha_3 = 12$

Problem 6:
At the Mathville post office, the largest shipping container is a rectangle measuring 27 klogs by 22 klogs. Ivan has been commissioned to make a rectangular painting which he must ship at the Mathville post office. The painting must be exactly 30 klogs long, but Ivan can choose the width. How wide can he make the painting (in klogs) and still fit it into a shipping container?
Solution:

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$m\angle DGH = m\angle CFG$

$x^2 + y^2 = 900$ and $\dfrac{y}{x} = \dfrac{27-x}{22-y}$ Thus

$22y - y^2 = 27x - x^2$. Substituting and simplifying yields

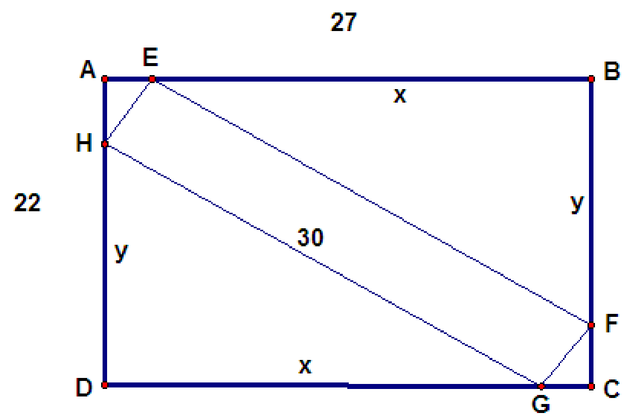
$900 + 22y - 2y^2 = 27\sqrt{900 - y^2}$.

Squaring both sides and simplifying

$4y^2 - 88y^3 - 2387y^2 + 39600y + 153900 = 0$

synthetic division

18	4	-88	-2387	39600	153000
		72	-288	-48150	-153900
	4	-16	-2675	-8550	0



Then $y = 18$ and $x = 24$ thus $CG = 3$ and $CF = 4$ and $FG = 5$ klogs

Problem 7:

Find $\left\lfloor \log_2\left(\dfrac{255!}{3!253!} + \dfrac{255!}{5!251!} + \dfrac{255!}{7!249!} + \ldots + \dfrac{255!}{253!3!}\right)\right\rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to ***x***.

Solution:

Let $N = \dfrac{255!}{3!253!} + \dfrac{255!}{5!251!} + \dfrac{255!}{7!249!} + \ldots + \dfrac{255!}{253!3!}$. Then

$$256(N+2) = \frac{256!}{1!255!} + \frac{256!}{3!253!} + \frac{256!}{5!251!} + \ldots + \frac{256!}{253!3!} + \frac{256!}{255!1!} = \binom{256}{1} + \binom{256}{3} + \ldots + \binom{256}{255}$$

Now we can write

$$(1+1)^{256} = \binom{256}{0} + \binom{256}{1} + \binom{256}{2} + \binom{256}{3} + \ldots + \binom{256}{255} + \binom{256}{256}$$ and also

$$(1-1)^{256} = \binom{256}{0} - \binom{256}{1} + \binom{256}{2} - \binom{256}{3} + \ldots - \binom{256}{256}$$

Subtracting the bottom from the top gives $2^{256} - 0^{256} = 2\cdot\left[\binom{256}{1} + \binom{256}{3} + \ldots + \binom{256}{255}\right]$

from which we see that $\binom{256}{1} + \binom{256}{3} + \ldots + \binom{256}{255} = 2^{255}$.

Hence $256(N+2) = 2^{255}$ so

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$N+2=2^{247}$ and $N=2^{247}-2$ and finally $\lfloor \log_2(N) \rfloor = 246$ since $2^{246} < N < 2^{247}$

Problem 8:
Find the sum of all possible digits *a* with the property that some cube of an integer ends in the digits *aaa* (in base 10).

Solution: We have examples showing that 0, 1, 3, 7, 8 and 9 are possible.
These were found via congruences and the binomial theorem. For example, for $a=3$, the units digit of the cube must be 7 as $7^3=343$ is the only cube ending is 3. Then $(10b+7)^3 \equiv 33 \bmod 100$ forces $b=7$ by expanding with the binomial theorem. Similarly, expanding $(100c+77)^3 \equiv 33 \bmod 1000$ shows that $c=4$.

To show that the other digits are not possible, let *n* be the cube ending *aaa*.
If $a=2,4, or\, 6$; then *n* must be even, hence it must be a multiple of 8 as it is a cube. However, any number ending in 222, 444, or 666 is not divisible by 8.
If $a=5$, then n must be divisible by 5, hence by 125. But no number ending in 555 is divisible even 125 (or even by 25).

Examples are as follows:

$10^3=1000$

$471^3=104487111$

$477^3=108531333$

$753^3=426957777$

$948^3=835896888$

$999^3=997002999$

The answer is the sum of the digits: 28

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