Talent Search Test 2 Solutions 2012

1) An ultraprime is a prime number all of whose (base-10) digits are prime numbers, and such that each of the integers obtained by rearranging its digits is also a prime number. Find the number of ultraprimes less than 1000.

Solution: The single-digit ultraprimes are 2, 3, 5, and 7. An ultraprime with more than one digit can contain only the digits 3 and 7, since if there is a 2 or 5, then any rearrangement with the 2 or 5 as the units digit cannot be prime. Any number all of whose digits are the same is also not prime. Of the remaining cases, 37 and 73 are both prime, as are 337, 373, and 733, while 377 is divisible by 13. Therefore there are a total of 9 ultraprimes less than 1000. (Note: there are no other ultraprimes less than 1 million; an integer cannot be ultraprime if its number of 7's is divisible by 3, or if its numbers of 3's and 7's are both even. Of the remaining possibilities, one can verify that 3377, 33337, 33737, 37777, 333373, and 377777 are all composite.)

2) In a mathematical competition, a contestant can score 5, 4, 3, 2, 1, or 0 points for each problem. Find the number of ways a contestant can score a total of 30 points for 7 problems.

Solution: The possibilities are summarized in the table below:

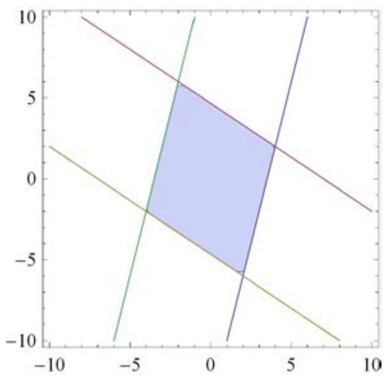
Score	5	4	3	2	1	0	Number of ways 7 problems can be scored
Number of problems given score	6	0	0	0	0	1	$7 = 7$
	5	1	0	0	1	0	$7 \cdot 6 = 42$
	5	0	1	1	0	0	$7 \cdot 6 = 42$
	4	2	0	1	0	0	$\binom{7}{4}\binom{3}{2} = 105$
	4	1	2	0	0	0	$\binom{7}{4}\binom{3}{1} = 105$
	3	3	1	0	0	0	$\binom{7}{3}\binom{4}{3} = 140$
	2	5	0	0	0	0	$\binom{7}{2} = 21$
Total number of possibilities:							462

3) Find the total area enclosed by the graph of $|3x + y| + |x - 2y| = 14$.

Solution: The region enclosed by the graph is defined by the inequality $|3x + y| + |x - 2y| \leq 14$.

The boundary of this region consists of portions of the four lines

$$l_1: +(3x + y) + (x - 2y) = 14,$$
$$l_2: +(3x + y) - (x - 2y) = 14,$$
$$l_3: -(3x + y) - (x - 2y) = 14$$
$$l_4: -(3x + y) + (x - 2y) = 14$$



Graphing these lines shows that the region is convex and has boundary points (in counterclockwise order) given by the intersections $l_1 \cap l_2$, $l_2 \cap l_3$, $l_3 \cap l_4$, $l_4 \cap l_1$.

The intersection of l_1 and l_2 satisfies 3x + y = 14, x - 2y =0, so (x, y) = (4, 2).
The intersection of l_2 and l_3 satisfies 3x + y = 0, x - 2y = -14, so (x, y) = (-2, 6).
The intersection of l_3 and l_4 satisfies 3x + y = -14, x - 2y = 0, so (x, y) = (-4, -2).
The intersection of l_4 and l_1 satisfies 3x + y = 0, x - 2y = 14, so (x, y) = (2, -6).

Then the area of the region is the area of the rectangle with vertices $(\pm 4, \pm 6)$ minus two triangles of area $\frac{1}{2} \cdot 4 \cdot 6$ and two triangles of area $\frac{1}{2} \cdot 2 \cdot 8$. Thus the area of the region is $8 \cdot 12 - 4 \cdot 6 - 2 \cdot 8 = 56$.

4) Rods 1 meter in length are used to build a rigid cubic framework. Twelve rods are needed to build a cube of side 1 meter. By fitting together eight of these unit cubes, a cubic framework can be constructed that has side 2 meters. By using 27 of the unit cubes, a cubic framework can be constructed that has side 3 meters. Each time the unit cubes are combined, there are duplicate rods along the edges where the cubes fit together. The duplicates may be removed (leaving one rod where there were previously two) and re-used elsewhere. What is the minimum necessary side length of the cubic structure such that the total length of all the rods used would stretch the 384,000 km from the Earth to the Moon?

Solution: We can see by induction that the following will be true.

Number of Cubes	Dimensions	Number of Rods
1	1 x 1 x 1	12=(1)(3)(4)
2	2 x 2 x 2	54=(2)(3)(9)
3	3 x 3 x 3	144=(3)(3)(16)
n	n x n x n	$3n(n+1)^2 = (n)(3)(n+1)^2$

Alternatively, if we align the rods in a 3-dimensional xyz coordinate system, then for each of the three directions x, y, and z, there are $n(n+1)(n+1)$ rods pointing in that direction, because there are $n+1$ layers each containing $n(n+1)$ rods. Solving for the total number of rods yields;

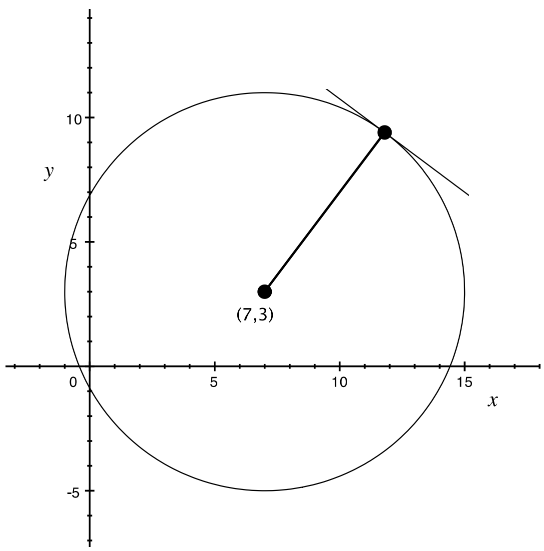
$$3n(n+1)^2 = 384{,}000{,}000$$

$$n \approx 503.3$$

So the minimum side length is 504 meters.

5) Suppose $x^2 + y^2 = 14x + 6y + 6$. What is the maximum value of $3x + 4y$?

Solution: We have quadratics in both x and y so we first complete the square on these terms. From this we see that the set of points (x, y) that satisfy the given equation also satisfy the equation $(x^2 - 14x + 49) + (y^2 - 6y + 9) = 6 + 49 + 9$ that is $(x-7)^2 + (y-3)^2 = 64$. These are the points on the circle with radius 8 centered at (7, 3). For a given constant c, the set of points that satisfy $3x + 4y = c$, or equivalently, $y = -\frac{3}{4}x + \frac{c}{4}$, lie on the line with slope -3/4 and y-intercept $c/4$. So, considered geometrically, we need to find the line with slope -3/4 that intersects the circle and has the largest value for its y-intercept.



This line will be tangent to the curve, as shown in the figure above, and the line segment that is the radius to this tangent point has slope 4/3. If we label the tangent point as $(7+3a, 3+4a)$, then we have

$$64 = (3a)^2 + (4a)^2 = 25a^2 \quad \Longrightarrow \quad a = \frac{8}{5}$$

So the point that gives the maximum value of $3x + 4y$ is $\left(7 + \frac{24}{5}, 3 + \frac{32}{5}\right) = \left(\frac{59}{5}, \frac{47}{5}\right)$, and this maximum value is

$$3 \cdot \frac{59}{5} + 4 \cdot \frac{47}{5} = \frac{177 + 188}{5} = \frac{365}{5} = 73$$

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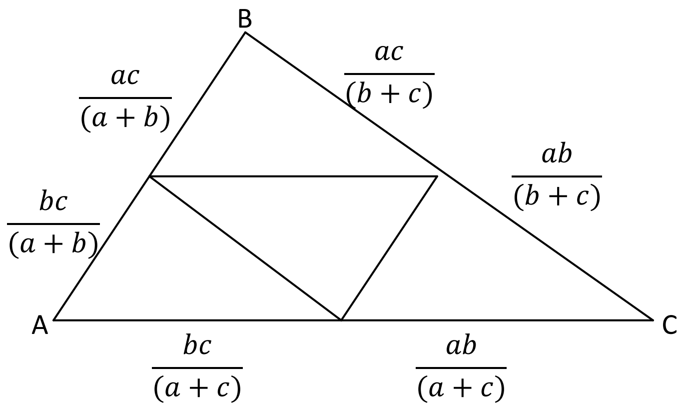
6) In triangle ABC, AB = 3, BC = 4, and AC = 6. D is the intersection of the angle bisector of A with BC, E is the intersection of the angle bisector of B with AC, and F is the intersection of the angle bisector of C with AB. Find the ratio of the area of triangle DEF to the area of triangle ABC.

Solution: In place of 3, 4, and 6, let the side lengths of triangle ABC be *a*, *b*, *c*. By the angle bisector theorem, the bisector to the side of length *a* divides that side into pieces in the ratio *b* : *c*. Therefore, we have a diagram as below:



The top triangle has area $\frac{a}{a+b}\cdot\frac{c}{b+c}$ times that of the full triangle, since its area is $\frac{1}{2}\cdot\frac{ac}{a+b}\cdot\frac{ac}{b+c}\cdot\sin(B)$ and the area of the full triangle is $\frac{1}{2}\cdot a\cdot c\cdot\sin(B)$. By the same argument we see that the fraction of the total taken up by the three outer triangles is $\frac{a}{a+b}\cdot\frac{c}{b+c}+\frac{b}{a+b}\cdot\frac{c}{a+c}+\frac{a}{a+c}\cdot\frac{b}{b+c}$, so the remaining fraction is $1-\frac{a}{a+b}\cdot\frac{c}{b+c}-\frac{b}{a+b}\cdot\frac{c}{a+c}-\frac{a}{a+c}\cdot\frac{b}{b+c}=1-\frac{3}{7}\cdot\frac{6}{10}-\frac{4}{7}\cdot\frac{6}{9}-\frac{3}{9}\cdot\frac{4}{10}=\frac{8}{35}$.

7) How many integers between 1,000,000 and 10,000,000 have digits which do not decrease in order from left to right? (1,234,567 and 1,122,599 would be such integers, but 1204567 and 1122595 would not).

Solution: Since 10,000,000 does not work, any such integer has seven digits, so let the integer be

N = abcdefg , where $1\le a\le b\le c\le d\le e\le f\le g\le 9$.

Now let a' = a, b' = b + 1, c' = c + 2, ... , and g ' = g + 6: then $1\le a'\le b'\le c'\le d'\le e'\le f'\le g'\le 15$.

Thus, every integer satisfying the conditions of the problem will give a 7-tuple of strictly increasing integers $(a',b',\dots,g')$ between 1 and 15 inclusive. We claim that, conversely, any 7-tuple of increasing integers between 1 and 15 inclusive, thought of as a 7-tuple $(a',b',\dots,g')$, will give rise to an integer $(a,b,\dots,g)$ satisfying the conditions of the problem. To see this, suppose we have such a 7-tuple $(a',b',\dots,g')$ with $1\le a'\le b'\le c'\le d'\le e'\le f'\le g'\le 15$. Since the a', b', ..., g ' are strictly increasing integers, the values a = a', b = b' - 1, ... , g = g ' - 6 are non-decreasing integers, and since

$a' \geq 1$ we see that they are all at least 1. Now since $g' \leq 15$ we see that $f' \leq 14, e' \leq 13, d' \leq 12, c' \leq 11, b' \leq 10, and\ a' \leq 9$ so that $g \leq 9, f \leq 9, e \leq 9, d \leq 9, c \leq 9, b \leq 9, a \leq 9$. So we see that each of a, b, c, d, e, f , g is a digit between 1 and 9, and they are non-decreasing. This is exactly the desired property.

Therefore, the number of 7-digit integers whose digits do not decrease is the same as the number of ways of choosing 7 integers from $\{1,2,\ldots,15\}$ which is $\binom{15}{7} = \frac{15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = 13\cdot11\cdot5\cdot9 = 6{,}435.$

8) The function $f(x)$ has the properties that $(i)\ f\left(\frac{x}{x+1}\right) = \frac{1}{2}f(x),$ and $(ii)\ f(1-x) = 1 - f(x)$, for all positive real numbers x. Find;

$$\sum_{k=1}^{\infty} f(\frac{1}{k})$$

Solution: Setting $x = \frac{1}{2}$ in (ii) yields $f\left(\frac{1}{2}\right) = 1 - f\left(\frac{1}{2}\right)\ so\ f\left(\frac{1}{2}\right) = \frac{1}{2}$. For $n > 1$, setting $x = \frac{1}{n-1}$ in (i) yields $\frac{1}{2}f\left(\frac{1}{n-1}\right) = f\left(\frac{\frac{1}{n-1}}{1+\frac{1}{n-1}}\right) = f(\frac{1}{n})$. Since $f\left(\frac{1}{2}\right) = \frac{1}{2}$ the following can be deduced;

n=2	$\frac{1}{2}f\left(\frac{1}{2-1}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}$	$f(1) = 1$
n=3	$\frac{1}{2}f\left(\frac{1}{3-1}\right) = f\left(\frac{1}{3}\right)$	$f\left(\frac{1}{3}\right) = \frac{1}{4}$
n=4	$\frac{1}{2}f\left(\frac{1}{4-1}\right) = f\left(\frac{1}{4}\right)$	$f\left(\frac{1}{4}\right) = \frac{1}{8}$
n=n	$\frac{1}{2}f\left(\frac{1}{n-1}\right) = f(\frac{1}{n})$	$f\left(\frac{1}{n}\right) = 2\cdot2^{-n}$

it follows that $f\left(\frac{1}{n}\right) = 2\cdot2^{-n}\ for\ n \geq 1.$

It follows by induction that,

$$\sum_{k=1}^{\infty} f(\frac{1}{k}) = \sum_{k=1}^{\infty} 2\cdot2^{-n} = 2\cdot\frac{\frac{1}{2}}{1-\frac{1}{2}} = 2$$