1) Find all real numbers x such that $(x^2 - x - 1)^{x^3+2x^2-9x-18} = 1$.

Answer: $x = -3, -2, -1, 0, 1, 2, 3$

Solution: An expression a^b is equal to 1 when $(i) a = 1, (ii) b = 0 \; and \; a > 0, or \; (iii) a - -1 \; and \; b$ is an even integer.

Case (i): $x^2 - x - 1 = 1$ yields $(x-2)(x+1) = 0, so \; x = 2 \; or \; x = -1$

Case (ii): Factoring $x^3 + 2x^2 - 9x - 18 = 0$ yields $(x+3)(x-3)(x+2) = 0, so \; x = 3, -3, -2$. Each of these values makes the base positive, so all work.

Case (iii): $x^2 - x - 1 = -1$ yields $x(x-1) = 0$, so possible values are $x = 0$ and $x = 1$. For both values of *x*, the exponent is even, so both work.

2) There are 100 people in a room. Each person states his/her birth date (month and day, excluding February 29). Let *N* be the number of different birth dates. What is the expected value of *N to the nearest integer*? (Assume that the 365 birth dates are equally likely.)

Answer: 88

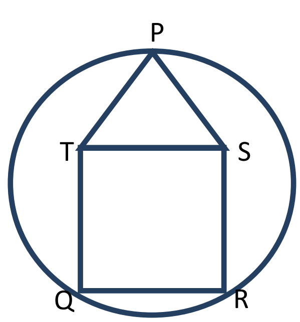
Solution: For $1 \le i \le 365$, let $X_i = 1$ if date *i* is among the stated birth dates and $X_i = 0$ otherwise. Then the expected number of different birth dates among the 100 people is:

$$E(X_1 + X_2 + X_3 + \cdots + X_{365}) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_{365}),$$

Because of linearity of expectation (independence of events is not necessary). Since all the dates are interchangeable, the sum on the right is equal to $365 \cdot E(X_1)$. The value of $E(X_1)$ is the probability that January 1 is among the 100 dates, which is the complementary probability of the event that January 1 is not one of the 100 dates. Hence $E(X_1) = 1 - \left({}^{364}\!/_{365}\right)^{100}$. Therefore, the expected number of different birth dates among 100 people is

$$365(1 - \left({}^{364}\!/_{365}\right)^{100}) \approx 88$$

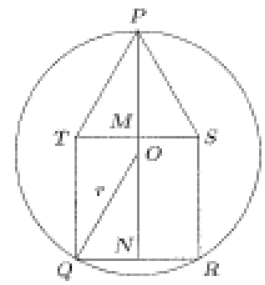
3) An equilateral triangle sits atop a square as in the diagram. All sides have length 1. A circle passes through points P, Q and R. What is the radius of the circle?



Answer: $r = 1$

Solution: Let *O* be the center of the circle, and let *r* be its radius. The perpendicular from point *P* to side *QR* bisects segments *TS* and *QR* at points *M* and *N*, respectively. We know that *O* lies on the segment *MN*, with $OM = r - PM = r - \sqrt{3}/2$ and $ON = \sqrt{r^2 - (1/2)^2}$. Since $OM + ON = 1$, we obtain the equation

$r - \frac{\sqrt{3}}{2} + \sqrt{r^2 - \frac{1}{4}} = 1$, solving for *r*:



$$\sqrt{r^2 - \frac{1}{4}} = 1 + \frac{\sqrt{3}}{2} - r$$

$$r^2 - \frac{1}{4} = \left(1 + \frac{\sqrt{3}}{2}\right)^2 - 2r\left(1 + \frac{\sqrt{3}}{2}\right) + r^2$$

$$r(2 + \sqrt{3}) = 1 + \sqrt{3} + \frac{3}{4} + \frac{1}{4}$$

From which we conclude that *r = 1*.

4) A set of positive integers is defined to be *wicked* if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a *wicked* set. Find the number of *wicked* subsets of the set {1,2,3,4,5,6,7,8,9,10}.

Answer: 504

Solution:

2. Let $S_n = \{1, 2, 3, \dots, n\}$ and W_n be the number of wicked subsets of S_n. We want to calculate W_{10}. We will proceed by looking at smaller wicked subsets and trying to construct larger ones. We start first with something easier.

   a. Consider $S_0 = \{\}$. There is only one subset – the empty set itself – and this subset is wicked; hence $W_0 = 1$.
   b. Consider $S_1 = \{1\}$. There are only two subsets, $\{\ \}$ and $\{1\}$, both of which are wicked; hence $W_1 = 2$.
   c. Consider $S_2 = \{1,2\}$. There are 4 subsets, each wicked; hence $W_2 = 4$.
   d. Consider S_3. There are 8 subsets, only one of which is not wicked (namely, $\{1,2,3\}$); hence $W_3 = 7$.
   e. Consider S_4. There are 16 subsets, only 3 of which are not wicked (namely, $\{1,2,3,4\}$, $\{1,2,3\}$, and $\{2,3,4\}$); hence $W_4 = 13$.

Now let A be any wicked subset of S_n. How many such sets *A* do not contain *n*? If *A* does not contain *n*, then *A* is itself a wicked subset of S_{n-1}. There are W_{n-1} such sets *A*.

Now assume *A* contains *n*. If *A* does not contain $n-1$, then the part of *A* that is left after removing *n* is a wicked subset of S_{n-2}. Thus there are W_{n-2} wicked subsets of S_n which contain *n* and do not contain $n-1$. If *A* does contain $n-1$, then *A* cannot contain $n-2$, since it cannot contain three consecutive integers. Hence, the part of *A* that we get after removing *n* and $n-1$ is a wicked subset of S_{n-3}. There are W_{n-3} such sets *A*.

These are all the possibilities for *A*. Thus,

$$W_n = W_{n-1} + W_{n-2} + W_{n-3}$$

(We can check that $W_3 = W_2 + W_1 + W_0$ and $W_4 = W_3 + W_2 + W_1$.) In other words, each term after the third is the sum of the three previous terms. Now we can write out the sequence starting at W_0 to get 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, …. Thus $W_{10} = 504$.

5) The "polar midpoint" of a segment whose endpoints are $A = (r_1, \theta_1)$ and $B = (r_2, \theta_2)$ when written in polar coordinates (with $0 \le \theta_1, \theta_2 < 2\pi$) is the point $C = (\frac{r_2+r_2}{2}, \frac{\theta_1+\theta_2}{2})$ in polar coordinates. Find the square of the distance between the (standard) midpoint and the polar midpoint of the segment whose coordinates in polar are $\left(4\sqrt{2}, \frac{\pi}{12}\right) and \left(8, \frac{7\pi}{12}\right)$.

Answer: $16 - 8\sqrt{2}$

Solution: The polar coordinates of the polar midpoint are $(2\sqrt{2}+4, \frac{\pi}{3})$. Therefore, in rectangular, the first point is $\left(4\sqrt{2}cos\frac{\pi}{12}, 4\sqrt{2}sin\frac{\pi}{12}\right) = \left(2\sqrt{3}+2, 2\sqrt{3}-2\right)$, the second point is

$\left(8cos\frac{7\pi}{12}, 8sin\frac{7\pi}{12}\right) = \left(2\sqrt{2}-2\sqrt{6}, 2\sqrt{2}+2\sqrt{6}\right)$, and the polar midpoint is

$\left(\left(2\sqrt{2}+4\right)cos\frac{\pi}{3}, \left(2\sqrt{2}+4\right)sin\frac{\pi}{3}\right) = \left(\sqrt{2}+2, 2\sqrt{3}+\sqrt{6}\right)$.

The midpoints coordinates are therefore $\left(1+\sqrt{2}+\sqrt{3}-\sqrt{6}, -1+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)$. So the square of its distance to the polar midpoint is

$$\left(-\sqrt{3}+\sqrt{6}+1\right)^2 + \left(\sqrt{3}-\sqrt{2}+1\right)^2 = 16 - 8\sqrt{2}.$$

6) A Morgan horse farmer willed that, upon his death, his three sons would receive the u-th, v-th, and w-th parts of his horses, respectively, where u, v, and w are integers. The three sons, however, could not evenly divide the team of horses according to the requirements, so they asked a Talent Search competitor for help. She rode over on her own horse, which she added to the team. It was then possible

to divide up the new team of horses into 4 parts: one u-th of the horses, one v-th of the horses, one w-th of the horses, and the Talent Search competitor's original horse.
Find all solutions (u, v, w, N).

Answer: 12 possible teams.

Solution. We seek an integer $N \geq 3$ and positive integers *u, v,* and *w* such that *u, v*, and *w* each divide $N+1$ (but not *N*), and $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)(N+1) = N$, or equivalently,

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w} = \frac{N}{N+1}.$$

Without loss of generality, we may assume $u \leq v \leq w$. The assumption $u \geq 5$ would yield $\frac{N}{N+1} \leq \frac{3}{5}$, contradicting $N \geq 3$. Thus $2 \leq u \leq 4$.

- If $u = 4$, then $\frac{1}{v}+\frac{1}{w} = \frac{3N-1}{4N+4}$ and, arguing as above, we see that $v \geq 5$ is impossible. Since $v \geq u$, we must have $v = 4$. Hence $\frac{1}{w} = \frac{N-1}{2N+2}$ and $w = \frac{2N+2}{N-1} = 2 + \frac{4}{N-1}$. It now follows that $N-1$ divides 4. Therefore, $N = 3$ or $N = 5$. Now $N = 5$ leads to $w = 3 < v$, a contradiction. Thus $N = 3$ and $w = 4$, which yields $(u, v, w, N) = (4,4,4,3)$, a suitable answer.

- If $u = 3$, a similar argument shows that we must have $v = 4$ or $v = 3$. As above, $v = 4$ leads to $w = \frac{12N+12}{5N-7}$, implying that $5N-7$ must divide $12N+12$. Then $5N-7$ must divide $5(12N+12) - 12(5N-7) = 144$. Since $N+1$ is also a multiple of both 3 and 4, we see that $N \geq 11$. It is easily seen that the only possibility is $5N-7 = 48$ and $N = 11$. But this gives $w = 3 < v$, a contradiction. Similarly, $v = 3$ leads to $N = 5$ or 11, and to the solutions $(3,3,6,5)$ and $(3,3,4,11)$.

- If $u = 2$, the same method calls for the examination of the cases $v = 6, 5, 4,$ or 3 and leads to 9 more solutions.

In conclusion, there are 12 solutions which are displayed in the chart below.

N	(u, v, w)
3	(4,4,4)
5	(2,6,6)
5	(3,3,6)
7	(2,4,8)
9	(2,5,5)
11	(2,3,12)
11	(2,4,6)
11	(3,3,4)
17	(2,3,9)
19	(2,4,5)
23	(2,3,8)
41	(2,3,7)

7) Suppose that *x, y,* and *z* are positive real numbers (none of which equals 1) such that

$\log_x yz + \log_y z = 5$ and $\log_z xy + \log_y x = 3$. Find all possible real numbers α such that $yz = x^\alpha$.

Answer: $\alpha = 2, {}^9/_2$.

Solution: Let $y = x^p$ and $z = x^q$, so that $z = y^{q/p}$. Then $log_x yz = p + q, log_y x = \frac{1}{p}, log_y z = \frac{q}{p}, log_z xy = \frac{1+p}{q}$. Hence the two equations become $p + q + \frac{q}{p} = 5$ and $\frac{1+p}{q} + \frac{1}{p} = 3$.

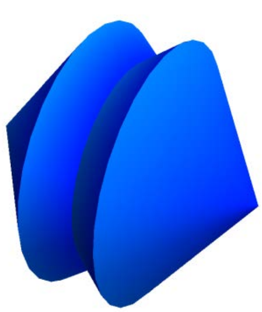
Clearing denominators yields $p^2 + pq + q = 5p\ and\ p + q + p^2 = 3pq$. Solving both equations for $\frac{q}{p}$ yields $\frac{5-p}{1+p} = \frac{q}{p} = \frac{1+p}{3p-1}$. Setting equal and cross-multiplying gives

$(5 - p)(3p - 1) = (1 + p)(1 + p), or\ 4p^2 - 14p + 6 = 0$. Solving gives $p = \frac{1}{2}, 3$ so that $(p, q) = \left(\frac{1}{2}, \frac{3}{2}\right), \left(3, \frac{3}{2}\right)$. So we obtain the two possibilities $y = x^{1/2}, z = x^{3/2}, and\ y = x^3, z = x^{3/2}$. Then

$yz = x^\alpha$ gives $\alpha = 2\ and\ \alpha = \frac{9}{2}$.

Remark: for both of these cases, solutions do exist: in the first case we can take $(x, y, z) = (4,2,8)$, and in the second case we can take $(x, y, z) = (4,64,8)$.

8) A certain credit card has the shape of a rectangle of dimensions 3 units x 4 units. If you rotate the card about one of its diagonals, what is the volume of the resulting solid of revolution? Here is a picture of the solid.

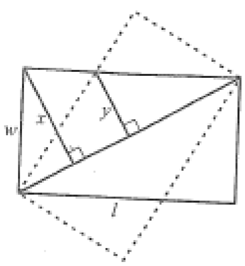


Answer: $Volume = \frac{\pi w^2(7l^4 - 2l^2w^2 - w^4)}{12l^2\sqrt{w^2+l^2}} = \frac{4269\pi}{320}$

Solution:

We solve the general problem for a rectangular card of width w and length l, where $w \leq l$.

Consider the diagram below.



Using the formula for the volume of a cone $\left(\frac{1}{3} base \cdot height\right)$, we have

$$volume = 2\left(\frac{1}{3}\pi x^2\sqrt{w^2+l^2} - \frac{1}{3}\pi y^2 \frac{\sqrt{w^2+l^2}}{2}\right)$$

$$volume = \frac{\pi}{3}\sqrt{w^2+l^2}(2x^2 - y^2)$$

By similar triangles, $x/w = l/\sqrt{w^2+l^2}$, and so $x^2 = \frac{w^2l^2}{(w^2+l^2)}$. Likewise, $y\Big/\left(\sqrt{w^2+l^2}/2\right) = w/l$, and so $y^2 = w^2(w^2+l^2)/(4l^2)$. Substituting in the expressions for $x^2 and\ y^2$, (where $w = 3\ and\ l = 4$) and simplifying, we obtain,

$Volume = \frac{\pi w^2(7l^4 - 2l^2w^2 - w^4)}{12l^2\sqrt{w^2+l^2}} = \frac{4269\pi}{320}$ cubic units.