Talent Search Test 1 Solutions 2013

1) In the array, two letters are called neighboring letters if they are adjacent to each other horizontally, vertically, or diagonally. Starting from any letter "M" on the outside of the array find the number of ways of spelling "MATH" by moving only between neighboring letters.

Answer: 104

M	M	M	M	M	M	M
M	A	A	A	A	A	M
M	A	T	T	T	A	M
M	A	T	H	T	A	M
M	A	T	T	T	A	M
M	A	A	A	A	A	M
M	M	M	M	M	M	M

Solution: The four corner A's have 5 adjacent M's and 1 adjacent T giving us 4*5 = 20

The eight A's adjacent to a corner A have 3 M's and 2 T's adjacent, hence 8*3*2=48

The four middle A's have 3 M's and 3 T's adjacent given 4*3*3 = 36

20+48+36 = 104

2) Find the number of positive integers n such that n plus the product of the (base-10) digits of n equals 2014.

Answer: 5 ($n = 1678, 1854,1933,1942,2014$)

Solution: Clearly $n \leq 2014$. If $2000 \leq n$ then the product of the digits is 0, so $n = 2014$.

Suppose that $n < 2000$. If $n < 1600$ then the product of the digits of n is at most $1 \cdot 5 \cdot 9 \cdot 9 = 405$, which does not give a large enough sum.

If $n = 16ab$ then we have $6ab + 1600 + 10a + b = 2012$, so that $b = \frac{414-10a}{6a+1}$. The only value of a which makes b a digit is $a = 7$, making $b = 8$ and giving $n = 1678$.

If $n = 17ab$ then we have $7ab + 1700 + 10a + b = 2012$, so that $b = \frac{314-10a}{7a+1}$. No value of a makes b an allowable digit.

If $n = 18ab$ then we have $8ab + 1800 + 10a + b = 2012$, so that $b = \frac{214-10a}{8a+1}$. The only value of a which makes b a digit is $a = 5$, making $b = 4$ and giving $n = 1854$.

If $n = 19ab$ then we have $9ab + 1800 + 10a + b = 2014$, so that $b = \frac{114-10a}{9a+1}$. The values of a which make b a digit is $a = 3$ with $b = 3$, and $a = 4$ with $b = 2$. These give $n = 1933$ and $n = 1942$ respectively.

3) The Calgary Lames and the Boston Ruins play a best-of-7 series with no ties, in which the series ends once one team has won exactly 4 games. The probability that the Ruins win any particular game is constant, and the games are independent. If it is exactly 45% more likely that the series lasts for 6 games than for 7 games, find the probability that it lasts for 4 games.

Answer: ${}^{641}/_{2401}$

Solution: Let p be the probability that the Ruins win any particular game. We then can observe that the series lasts for k games, for $4 \le k \le 7$, precisely when the team that wins the last game won exactly 3 of the first $k-1$ games, this is true because the winning team's fourth win must occur in the kth game. Thus, there are $\binom{k-1}{3}$ ways of choosing where the wins and losses occur in the first $k-1$ games. Hence, the probability that the series lasts 6 games is $\binom{5}{3}[p^4(1-p)^2 + p^2(1-p)^4] = 10p^2(1-p)^2(2p^2-2p+1)$ and the probability that the series lasts 7 games is $\binom{6}{3}[p^4(1-p)^3 + p^3(1-p)^4] = 20p^3(1-p)^3$. Hence the given information implies $\frac{10p^2(1-p)^2(2p^2-2p+1)}{20p^3(1-p)^3} = 1.45$; cancelling yields $\frac{2p^2-2p+1}{2p-2p^2} = \frac{29}{20}$.

Setting $y = 2p - 2p^2$ yields $\frac{1+y}{y} = -\frac{29}{20}$ so $y = -\frac{20}{49}$. Then we have $2p - 2p^2 + \frac{20}{49} = 0$, and factoring gives $2\left(p - \frac{5}{7}\right)\left(p - \frac{2}{7}\right) = 0$, so $p = \frac{2}{7} or \frac{5}{7}$.

4) a) Find all $x > 0$ such that $x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cdots}}} = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$ Answer: x=1

Let $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cdots}}} = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$. Then from the first equation, we have $y = x + \frac{1}{y}$, so $y^2 = xy + 1$. From the second equation, we have $y = \sqrt{x + y}$ so $y^2 = x + y$. Thus, $x + y = xy + 1$ and $xy - x - y + 1 = 0$. If $x = 1$ then both equations give $y^2 - y - 1 = 0$ so that $y = \frac{1+\sqrt{5}}{2}$ (since the minus sign is not allowed because $y > 0$). If $y = 1$ then we would obtain $x = 0$ but this is not allowed. Hence the answer is $x = 1$.

b) Find all $x > 0$ such that $x - \cfrac{1}{x - \cfrac{1}{x - \cfrac{1}{x - \cdots}}} = \sqrt{x - \sqrt{x - \sqrt{x - \sqrt{x - \cdots}}}}$ Answer: x=2

Let $z = x - \cfrac{1}{x - \cfrac{1}{x - \cfrac{1}{x - \cdots}}} = \sqrt{x - \sqrt{x - \sqrt{x - \sqrt{x - \cdots}}}}$. Then from the first equation, we have $z = x - \frac{1}{z}$, so $z^2 = xz - 1$. From the second equation, we have $z = \sqrt{x - z}$ so $z^2 = x - z$. Thus, $x - z = xz - 1$ and $xz - x + z - 1 = 0$, or $(x+1)(z-1) = 0$. Since x is positive we must have $z = 1$; then both equations give $x = 2$ which is the only solution.

5) Find all pairs of non-zero real numbers (x, y) that are solutions to the simultaneous equations $x^{x+y} = y^3$ and $y^{x+y} = x^6y^3$.

Answer: $(2,4), (-3,9), (1,1), \left(\frac{1}{2}(-3+\sqrt{5}), \frac{1}{2}(-3-\sqrt{5})\right), \left(\frac{1}{2}(-3-\sqrt{5}), \frac{1}{2}(-3+\sqrt{5})\right), (-1,1)$

Solution: If either $x = 0$ or $y = 0$, the other is equal to 0 also, but then 0^0 is undefined. Multiplying $x^{x+y} = y^3$ and $y^{x+y} = x^6y^3$, we have $(xy)^{x+y} = (xy)^6$.

We may have $x + y = 6$. Substituting back into $x^{x+y} = y^3$, we have $x^6 = y^3$ or $y = x^2$. From $x + x^2 = 6$, we have $0 = x^2 + x - 6 = (x - 2)(x + 3)$. If $x = 2, y = 4$, and if $x = -3, y = 9$.

We may also have $xy = 1$. Then $y = \frac{1}{x}$ and $x^{x+\frac{1}{x}+3} = 1$. Either $x = 1$ or $x + \frac{1}{x} + 3 = 0$. The solutions to the latter equation, rewritten as $x^2 + 3x + 1 = 0$ are $x = \frac{1}{2}(-3 \pm \sqrt{5})$. If $x = 1, y = 1$. If $x = \frac{1}{2}(-3 + \sqrt{5}), y = \frac{1}{2}(-3 - \sqrt{5})$ and if $x = \frac{1}{2}(-3 - \sqrt{5}), y = \frac{1}{2}(-3 + \sqrt{5})$.

Finally, we may have $xy = -1$ with $x + y$ an even integer. Then $y = -\frac{1}{x}$ and $x^{x-\frac{1}{x}+3} = -1$. We must have $x = -1$ and thus $y = 1$.

So there are six solutions: $(2,4), (-3,9), (1,1), \left(\frac{1}{2}(-3 + \sqrt{5}), \frac{1}{2}(-3 - \sqrt{5})\right), \left(\frac{1}{2}(-3 - \sqrt{5}), \frac{1}{2}(-3 + \sqrt{5})\right), (-1,1)$.

6) Inside a circle of radius 5 is inscribed a trapezoid of height 4, one of whose bases is a diameter of the circle. A triangle having the property that each of its sides is parallel to a side of the trapezoid is also inscribed in the circle. Find the area of the triangle.

Answer: 32

Solution: Take the circle's center as the origin and let the vertices of the trapezoid be $(r, 0), (a, b), (-a, b), and\ (-r, 0)$; by the data given, $r = 5,\ a = 3, and\ b = 4$. The triangle is isosceles and has one vertex at $(0, r)$. One of the other vertices lies on the intersection of the circle and the line through $(0, r)$ parallel to the line determined by $(r, 0)$ and (a, b) i.e., on the line $y = \frac{a+r}{b}x + r$. Then it is clear that the point $(b, -a)$ lies on the line and on the circle, hence is one of the vertices of the triangle. The other vertex, by symmetry, is then $(-b, -a)$, so the triangle's area is $\frac{1}{2} \cdot 2b \cdot (a + r) = b(a + r) = 32$. Remark: The area of the trapezoid is also $b(a + r)$, so the triangle and trapezoid always have the same area.

7) Find the number of ordered pairs of real number (x, y) satisfying the equations $x^2 + y^2 = 2013$ and $(\tan \pi x)(\tan \pi y) = 1$.

Answer: 252

Solution: The second equation is equivalent to $\tan \pi x = \cot \pi y = \tan(\frac{\pi}{2} - \pi y)$, hence $\pi x - \left(\frac{\pi}{2} - \pi y\right) = k\pi$ for some integer k. Thus $x + y = k + \frac{1}{2}$. Plugging this into the first equation yields $x^2 + \left(k + \frac{1}{2} - x\right)^2 = 2013$ or after completing the square and rearranging; $2\left[x-\left(\frac{k}{2}+\frac{1}{4}\right)\right]^2+\frac{\left(k+\frac{1}{2}\right)^2}{2}=2013$. Or, $\left[x-\left(\frac{k}{2}+\frac{1}{4}\right)\right]^2+\frac{\left(k+\frac{1}{2}\right)^2}{4}=\frac{2013}{2}$. Hence $\frac{\left(k+\frac{1}{2}\right)^2}{4}\leq\frac{2013}{2}$, so that $-\sqrt{4026}-\frac{1}{2}\leq k\leq\sqrt{4026}+\frac{1}{2}$. Since k is an integer, it follows that $-63\leq k\leq 62$. Each value of k gives two possible pairs (x,y), since $2\left[x-\left(\frac{k}{2}+\frac{1}{4}\right)\right]^2+\frac{\left(k+\frac{1}{2}\right)^2}{2}=2013$ gives two values of x each of which then has y determined via $x+y=k+\frac{1}{2}$. Hence the total number of solutions is $2(126)=252$.

8) One angle in a triangle with sides $a-3$, $a-2$, and $a-1$ is supplementary to one angle in a triangle with sides $a-1,a,a+1$. Find all possible values of a.

Answer: $a=1+\sqrt{10},\frac{9+\sqrt{73}}{4},\frac{7+\sqrt{13}}{2}$

Solution: In order for both triangles to be non-degenerate, we require $a>4$: then we see that the triangle with sides $a-1,a,a+1$ is acute, since for $a>4$ we have $(a+1)^2<a^2+(a-1)^2$. Hence the other triangle must be obtuse, so one of the two supplementary angles must be the largest angle in the triangle with sides $a-3,a-2,a-1$. Now, in a triangle with sides $b-1,b,b+1$, the cosines of the three angles satisfy, respectively, $(b-1)^2=b^2+(b+1)^2-2b(b-1)\cos\theta_1$, $b^2=(b-1)^2+(b+1)^2-2(b-1)(b+1)\cos\theta_2$, and $(b+1)^2=(b-1)^2+b^2-2b(b-1)\cos\theta_3$. Solving yields $\cos\theta_1=\frac{b+4}{2(b+1)},\cos\theta_2=\frac{b^2+2}{2(b^2-1)}$,

$and\ \cos\theta_3=\frac{b-4}{2(b-1)}$. From the above formulas, we see that the cosine of the largest angle in the $a-3,a-2,a-1$ triangle is $\cos\propto=\frac{a-6}{2(a-3)}$. If $\propto+\theta=\pi$, then $\cos\propto=-\cos\theta$, so we require $-\frac{a-6}{2(a-3)}=\frac{a+4}{2(a+1)}$,

$-\frac{a-6}{2(a-3)}=\frac{a^2+2}{2(a^2+1)}$, or $-\frac{a-6}{2(a-3)}=\frac{a-4}{2(a-1)}$. Cross multiplying, cancelling, and rearranging yields $a^2-2a-9=0, a(2a^2-9a+1)=0, and\ a^2-7a+9=0$, respectively. Solving yields $a=1\pm\sqrt{10},\frac{9\pm\sqrt{73}}{4},\frac{7\pm\sqrt{13}}{2}$. Since we require $a>4$, we have three solutions: $a=1+\sqrt{10},\frac{9+\sqrt{73}}{4},\frac{7+\sqrt{13}}{2}$.