Talent Search Test 2 Solutions 2013

1) A survey of 350 math students was taken to see which Star Trek Captain was their favorite. The captains listed were Kirk, Picard and Archer.

- 46 students liked none of the captains listed (maybe preferring Sisko and Janeway)
- 1/3 of the students who liked Archer also liked Picard but not Kirk
- 98 students only liked Kirk
- 14 students liked all three captains
- 184 students liked Picard
- 56 students liked both Kirk and Picard
- 82 students liked more than 1 captain

Find the number of students who liked;

a) Only Archer

b) Only Picard

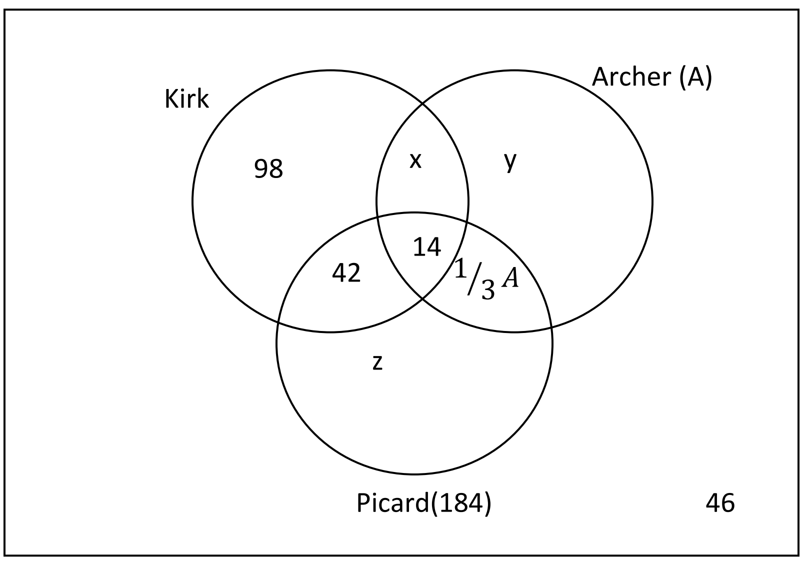
c) Kirk and Archer

Answers: a) 14

b) 110

c) 22

Solution:



56 students like Kirk and Picard, so 56 - 14 = 42.

Using the assigned Variables we get the following equations.

Equation (1) $56 + {}^1/_3\, A + z = 184 \; so \; {}^1/_3\, A + z = 128$

Equation (2) $98 + 56 + x + y + {}^1/_3\, A + z = 350 - 46$

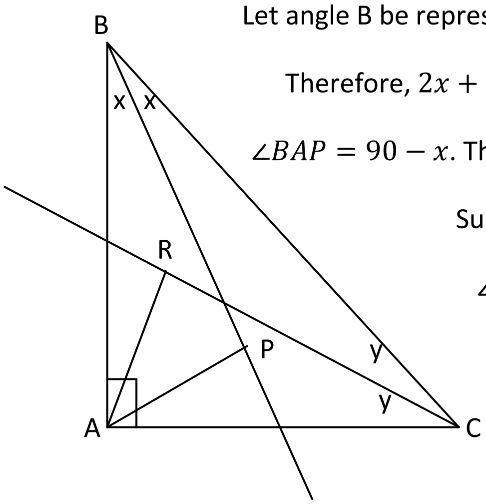
combining (1) and (2) we get $x + y = 22$

Equation (3) $x + y + {}^1/_3\, A + 14 = A$ substituting for $x + y$ from above, we find

$A = 54$ which then allows us to go back and solve for the remainder of the variables

2) Triangle ABC has $\angle BAC = 90^o$. The perpendiculars from A to the internal bisectors of $\angle ABC$ and $\angle ACB$ intersect said bisectors at R and P, respectively. Determine the measure of $\angle RAP$.

Solution: Answer: 45 degrees



Let angle B be represented by $2x$ and angle C by $2y$.

Therefore, $2x + 2y = 90^o$ and $x + y = 45^o$. $\angle RAC = 90^o - y$ and

$\angle BAP = 90 - x$. Therefore, $\angle RAP = \angle RAC - \angle PAC = \angle BAP - (\angle BAC - BAP)$.

Substituting in, $\angle RAP = 90^o - y - (90^o - (90^o - x))$

$\angle RAP = 90^o - (y + x) = 90^o - 45^o = 45^o$

3) The positive integer X has 4026 digits, all of which are 4's. The positive integer Y has 2013 digits, all of which are 8's. What is the sum of the digits of $\sqrt{X - Y}$? Answer: 12,078

Solution: $X - Y = 4(1 + 10 + 10^2 + \cdots + 10^{4025}) - 8(1 + 10 + 10^2 + \cdots + 10^{2012})$

$$X - Y = 4((10^{2013} + 10^{2014} + \cdots + 10^{4025}) - (1 + 10 + 10^2 + \cdots + 10^{2012}))$$

$$X - Y = 4(10^{2013} - 1)(1 + 10 + 10^2 + \cdots + 10^{2012})$$

$$X - Y = 4(10 - 1)(1 + 10 + 10^2 + \cdots + 10^{2012})^2$$

$$X - Y = (6(1 + 10 + 10^2 + \cdots + 10^{2012}))^2$$

Hence $\sqrt{X - Y}$ is a positive integer with 2013 digits all of which are 6's, or 6(2013)=12,078.

4) A check is cashed at a bank and mistakenly the teller pays out the number of cents as dollars and the number of dollars as cents. The person receiving the money spends \$3.50 before noticing the mistake, and on counting the remaining money finds it is exactly double the amount of the original check. What was the original check amount?

Answer: \$14.32

Solution:

Let d and c represent the number of dollars and cents on the check. The amount of the check is $100d + c$, where $c < 100$.

$$100c + d - 350 = 2(100d + c)$$

$$98c - 350 = 199d$$

$$14(7c - 25) = 199k$$

If $k = 1$, we get $c = 32$ and $d = 14$. For higher values of k we either get no integer solution or $c > 100$, which is not allowed.

5) Three positive integers are randomly chosen, with replacement, from the set $\{1,2,\ldots,2013\}$. Let t be the expected value of the smallest of the three integers. Find the integer closest to t.

Solution: Answer: 504

Let $P(k)$ be the probability that k is the smallest of the three integers, where we choose from the integers $\{1,2,\ldots,n\}$; here $n = 2013$. We want to find the value of $P(1) + 2P(2) + 3P(3) + \cdots + nP(n)$. Let $Q(k)$ be the probability that each of the integers is at least k: in such case, each of the integers must be chosen from $\{k, k+1, \ldots, n\}$, hence $Q(k) = \frac{(n+1-k)^3}{n^3}$. We can also see that $Q(k) = P(k) + P(k+1) + \cdots + P(n)$. Then $P(1) + 2P(2) + 3P(3) + \cdots nP(n) = Q(1) + Q(2) + \cdots Q(n) = \sum_{k=1}^{n} \frac{(n+1-k)^3}{n^3} = \sum_{j=1}^{n} \frac{j^3}{n^3} = \frac{(n+1)^2}{4n} = \frac{n+2}{4} + \frac{1}{4n}$. For $n = 2013$, the closest integer to this value is $\frac{2016}{4} = 504$.

6) In the multiplication problem below, each letter represents a different digit: Which digit does C represent?

$$\begin{array}{r} A\ B\ C\ D\ E\ F\ G\ H \\ \underline{\times \qquad\qquad\quad A\ J} \\ C\ C\ C\ C\ C\ C\ C\ C\ C \end{array}$$

Answer: 6

Solution:

Begin by factoring out C from $CCCCCCCCC$ and get $C \cdot 111111111 = C \cdot 9 \cdot 12345679$. So we have $ABCDEFGH \cdot AJ = C \cdot 9 \cdot 12345679 = k \cdot 12345679$. It can then be observed that $A = k$ for $k = 1,2,4$ and $A = k + 1$ for $k = 5,7,8$. The values of k provide distinct digits for $ABCDEFGH$. $k \cdot AJ = C \cdot 9$ and $k \cdot AJ \geq 10k^2$ and $9C \leq 81$. We see $k^2 \leq 8.1$ so $k = 1$ or 2. $k = 1$ does not work because $ABCDEFGH = 12345679$ yields $C = 3$, hence $AJ = 27$ contradicting $A = 1$. Trying $k = 2$ yields $24691358 \cdot 27 = 666666666$.

7) Triangle ABC is inscribed in circle O. Points D and E are chosen on AB and BC respectively, such that $AC = BD = 4$, $AD = BE = 2$, and $BC = 3$. Segment DE is extended to intersect arc AB at F and arc BC at G. Find $|DF - EG|$.

Solution: Answer: $\frac{9}{4}$

Let $m\angle ABC = \theta$. By the law of cosines in ABC, $4^2 = 6^2 + 3^2 - 2 \cdot 3 \cdot 6 \cdot \cos\theta$, so $\cos\theta = \frac{29}{36}$. Then by the law of cosines in BDE, $DE^2 = 4^2 + 2^2 - 2 \cdot 2 \cdot 4 \cdot \frac{29}{36} = \frac{64}{9}$, so $DE = \frac{8}{3}$. Now let $DF = x$ and $EG = y$. By the power-of-a-point theorem, we have $DF \cdot DG = DA \cdot DB$ and $EF \cdot EG = EB \cdot EC$, so $x \cdot \left(\frac{8}{3} + y\right) = 2 \cdot 4$ and $\left(x + \frac{8}{3}\right) \cdot y = 2 \cdot 1$; subtracting yields $\frac{8}{3} \cdot (x - y) = 6$ so $x - y = \frac{9}{4}$.

8) Find the value of the infinite product $3 \cdot 3^{\log_4 3} \cdot 3^{\log_4 3^{\log_4 3}} \cdot 3^{\log_4 3^{\log_4 3^{\log_4 3}}} \ldots$. Express your answer in the form $a^{\log_b c}$ for rational numbers a, b, and c.

Solution: Answer: $3^{\log_{4/3} 4}$

Let the value of the product be P. Then;

$$\log_3 P = \log_3[3] + \log_3\left[3^{\log_4 3}\right] + \log_3\left[3^{\log_4 3^{\log_4 3}}\right] + \log_3\left[3^{\log_4 3^{\log_4 3^{\log_4 3}}}\right] + \cdots$$

$$= 1 + \log_4 3 + \log_4 3^{\log_4 3} + \log_4 3^{\log_4 3^{\log_4 3}} + \cdots$$

$$= 1 + \log_4 3 + [\log_4 3]^2 + [\log_4 3]^3 + \cdots$$

Now this last sum is a convergent geometric series (since the common ratio is $\log_4 3$ which is positive and less than 1) and its value is $\frac{1}{1-\log_4 3} = \frac{1}{\log_4(4/3)} = \log_{4/3} 3$. Hence the answer is $3^{\log_{4/3} 4}$.

Note: The answer can be written in various ways in the given form: for example, one can write

$2^{\log_{4/3} 4} = 4^{\log_{4/3} 3} = 3^{\log_{4/3} 4} = 9^{\log_{4/3} 2}$.