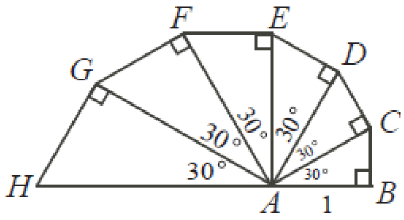
1) A shell is formed from six triangular sections, as shown. Each triangle has interior angles of 30^o, 60^o, and 90^o. If AB has a length of 1 cm, what is the length of AH, in cm?



Answer: $\frac{64}{27}$

Solution:

In a $30^o - 60^o - 90^o$ triangle the ratio of the side opposite the 90^o angle to the side opposite the 60^o angle is $2:\sqrt{3}$. Therefore:

$$\frac{AH}{AG}=\frac{AG}{AF}=\frac{AF}{AE}=\frac{AE}{AD}=\frac{AD}{AC}=\frac{AC}{AB}=\frac{2}{\sqrt{3}}$$

$$AH=\frac{2}{\sqrt{3}}AG=\left(\frac{2}{\sqrt{3}}\right)^2AF=\left(\frac{2}{\sqrt{3}}\right)^3AE=\left(\frac{2}{\sqrt{3}}\right)^4AD=\left(\frac{2}{\sqrt{3}}\right)^5AC=\left(\frac{2}{\sqrt{3}}\right)^6AB$$

Since $AB=1, AH=\left(\frac{2}{\sqrt{3}}\right)^6=\frac{64}{27}$

2) Find all integer solutions of the system of simultaneous equations:

$$Equation\ 1: ab+cd=-1$$

$$Equation\ 2: ac+bd=-1$$

$$Equation\ 3: ad+bc=-1$$

Answer: $a=\pm 2, b=c=d=\mp 1$

$b=\pm 2, a=c=d=\mp 1$

$c=\pm 2, a=b=d=\mp 1$

$d=\pm 2, a=b=c=\mp 1$

Solution:

Subtracting equation 2 from equation 1 we find $(a-d)(b-c)=0$ which is equivalent to either $a=d$ or $b=c$. If $b=c$ then the second and third equations can be rewritten as;

$$b(a+d)=-1$$

$$ad + b^2 = -1$$

Since b and $a + d$ are integers we have $b = \pm 1$, therefore $ad = -1 - b^2 = -2$. Four cases result,

$(a) a = \pm 2, d = \pm 1$; $(b) a = \pm 1, d = \mp 2$. In case (a) we have that $a + d = \pm 1$ and $c = b = \mp 1$. In case (b) we have $a + d = \mp 1$ and $c = b = \pm 1$. Similarly, if $a = d$, we obtain another four answers to get the final 8 solutions.

3) Two teams play a 7-game series, where the first team to win 4 games wins the series. If a team has won w games and lost I games so far in the series, its probability of winning the next game is ${(w+1)}/{(I+w+2)}$.

Find the probability that the series requires 7 games to decide the winner.

Answer: 1/7

Solution:

In order for the series to last 7 games, each team must win 3 of the first 6 games: there are $\binom{6}{3} = 20$ ways in which these wins can be arranged. For any of the 20 possible outcomes, consider the probability that the corresponding sequence of wins and losses occurs: it is $\frac{\blacksquare}{2} \cdot \frac{\blacksquare}{3} \cdot \frac{\blacksquare}{4} \cdot \frac{\blacksquare}{5} \cdot \frac{\blacksquare}{6} \cdot \frac{\blacksquare}{7}$, where the elements in the six boxes are the integers 1, 2, 3, (for the wins of team A) and 1, 2, 3 (for the wins of team B) in the corresponding order. Thus, for example, the probability that the wins belong to AABABB in that order is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{5} \cdot \frac{2}{6} \cdot \frac{3}{7}$, as can be verified by calculating the probability of each team winning the appropriate game in the series. Therefore, the probability that the series lasts 7 games is $\frac{6!}{3! \cdot 3!} \cdot \frac{3! \cdot 3!}{7!} = \frac{1}{7}$.

4) In triangle $ABC, \overline{AB} = 4, \overline{AC} = 5, \overline{BC} = 6$, and point D lies on $\overline{AC}$ with $\overline{CD} = 2$. Points E and F on $\overline{AB}$ and $\overline{BC}$ (respectively) are chosen so that when line segments $\overline{DE}$ and $\overline{DF}$ are drawn, they divide ABC into three regions of equal area. Find the length of segment $\overline{EF}$.

Answer: $\frac{5\sqrt{7}}{9}$

Solution:

We denote areas of regions using brackets. We have $[CDF] = \frac{1}{2}\overline{CD} \cdot \overline{CF} \cdot \sin(C)$, which by the given information is also equal to $\frac{1}{3}[ABC] = \frac{1}{3} \cdot \frac{1}{2} \cdot \overline{AC} \cdot \overline{BC} \cdot \sin(C)$. Equating and plugging in the given information yields $2 \cdot \overline{CF} = \frac{1}{3} \cdot 5 \cdot 6$, so $\overline{CF} = 5$. Similarly, we have $[ADE] = \frac{1}{2}\overline{AD} \cdot \overline{AE} \cdot \sin(A)$, which by the given information is also equal to $\frac{1}{3}[ABC] = \frac{1}{3} \cdot \frac{1}{2} \cdot \overline{AC} \cdot \overline{AB} \cdot \sin(A)$. Equating and plugging in the given information yields $3 \cdot \overline{AE} = \frac{1}{3} \cdot 5 \cdot 4$, so $\overline{AE} = \frac{20}{9}$. We then have $\overline{BE} = \frac{16}{9}$ and $\overline{BF} = 1$. Using the Law of Cosines in ABC we get $5^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cdot \cos(B)$, whence $\cos(B) = \frac{9}{16}$. Then the Law of Cosines in BEF yields $EF^2 = 1^2 + \left(\frac{16}{9}\right)^2 - 2 \cdot 1 \cdot \left(\frac{16}{9}\right) \cdot \left(\frac{9}{16}\right) = \frac{175}{81}$. Hence, $\overline{EF} = \frac{\sqrt{175}}{9} = \frac{5\sqrt{7}}{9}$.

5) Determine all real solutions to the system of equations.

$$x + \log(x) = y - 1$$

$$y + \log(y - 1) = z - 1$$

$$z + \log(z - 2) = x + 2$$

Answer (1,2,3)

Solution:

Let $a = x, b = y - 1, and\ c = z - 2$. The system becomes;

$$Equation\ (1) \quad a + \log(a) = b$$

$$Equation\ (2) \quad b + \log(b) = c$$

$$Equation\ 3 \quad c + \log(c) = a$$

which has a fairly obvious solution $(a, b, c) = (1,1,1)$, giving $(x, y, z) = (1,2,3)$. We claim this is the only solution. To see this first observe that if $a = 1$, then $b = 1$ and then $c = 1$. Next suppose $a < 1$. Then since $\log(a)$ is negative, equation 1 gives $b < a$. Since $a < 1$ we see that $b < 1$, meaning that $\log(b)$ is negative. Then equation 2 gives $c < b$, so, by applying the same logic, we see that $\log(c)$ is negative. Equation 3 then gives $a < c$. Combining all of the above inequalities yields $a < c < b < a$, which is impossible.

We apply the same logic in the event that $a > 1$: here, since $\log(a)$ is positive, equation 1 gives $b > a$. Then $\log(b)$ is positive so equation 2 implies $c > b$. Finally, $\log(c)$ is positive, so equation 3 implies $a > c$. Combining all inequalities gives $a > c > b > a$, which is likewise impossible.

6) The natural numbers a, b, c have the property that $a^3$is divisible by b; b^3 is divisible by c; and c^3 is divisible by a. Prove that $(a + b + c)^{13}$ is divisible by abc, but that $(a + b + c)^{12}$ need not be divisible by abc.

Solution:

Expanding the expression $(a + b + c)^{13}$ we obtain the sum of the terms of the type $a^i b^j c^k$ where $i, j, k \geq 0$ and $i + j + k = 13$.

Case 1: All of i, j, k are all positive: in this case,$a^i b^j c^k = abc(a^{i-1} b^{j-1} c^{k-1})$ is clearly divisible by abc.

Case 2: Two of i, j, k are zero: the hypotheses are symmetric, so by permuting the variables, without loss of generality we may suppose $j = k = 0$. Then $a^{13} = aa^3a^9$ is divisible by abc, because $a^3$is divisible by b and a^9 is divisible by b^3, which is in turn divisible by c.

Case 3: Exactly one of i, j, k are zero: the hypotheses are symmetric, so by permuting the variables, without loss of generality we may suppose $i = 0$. Then $j + k = 13$ and $1 \le j \le 12$. We split into further cases:

- If $1 \le j \le 3$, then $b^jc^k = c^9bc\left(b^{j-1}c^{3-j}\right)$ is divisible by abc, since c^9 is divisible by b^3 which is divisible by a.
- If $4 \le j \le 9$, then $b^jc^k = c^3bb^3\left(b^{j-4}c^{9-j}\right)$ is divisible by abc, since c^3 is divisible by a, and b^3 is divisible by c.
- If $10 \le j \le 12$, then $b^jc^k = b^9bc\left(b^{j-10}c^{12-j}\right)$ is divisible by abc, since b^9 is divisible by c^3 and c^3 is divisible by a.

To prove that $(a + b + c)^{12}$ is not divisible by abc you may use a counterexample below;

Let $a = p, b = p^3, c = p^9$ where p is a prime, $(a + b + c)^{12} = p^{12}(1 + p^2 + p^8)$ is not divisible by $abc = p^{13}$.