Vermont Mathematics Talent Search, Solutions to Test 2, 2014-2015

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1. Three numbers form a geometric sequence. The arithmetic mean of the first two is -9, and the arithmetic mean of the first and third terms is -15. Find the smallest possible value of the first term.

   Answer: -27.

   Solution: Let the three terms (in order) be a, ar, ar^2. The given information states $\dfrac{a+ar}{2} = -9$ and $\dfrac{a+ar^2}{2} = -15$.

   Rearranging gives $a(1+r) = -18$ and $a(1+r^2) = -30$. Solving the first equation for a yields $a = -\dfrac{18}{1+r}$, and plugging into the second equation yields $-\dfrac{18}{1+r}(1+r^2) = -30$, so $-18 - 18r^2 = -30r$.

   Rearranging yields $18r^2 - 30r + 18 = 0$, so factoring yields $6(3r+1)(r-2) = 0$. Then $(r, a) = (2, -6)$ or $(-1/3, -27)$. The smallest possible value for a is $\boxed{-27}$.

2. A circular sector has the same perimeter and area as a rectangle. (The perimeter of the sector includes the two radii.) Prove that the radius of the sector equals one of the side lengths of the rectangle.

   Solution 1: Let the rectangle have sides a and b, and the sector have radius r and arclength $2c$. Then we have $2a + 2b = 2r + 2c$ and $ab = rc$. This is equivalent to $a + b = r + c$ and $ab = rc$, so, if x is a variable, then we see $(x-a)(x-b) = (x-r)(x-c)$. Since a quadratic polynomial's roots are unique, we conclude either that $r = a$ or $r = b$.

   Solution 2: Let the rectangle have sides a and b, and let the sector have radius r and angle θ. Then $2a + 2b = 2r + r\theta$ and $ab = \dfrac{1}{2}r^2\theta$. Solving the first equation for θ yields $\theta = 2\dfrac{a+b-r}{r}$, and then plugging into the second equation yields $ab = \dfrac{1}{2}r^2 \cdot 2\dfrac{a+b-r}{r} = r(a+b-r)$. Rearranging and factoring yields $(r-a)(r-b) = 0$, so $r = a$ or $r = b$.

3. If Q is a convex quadrilateral whose four side lengths and two diagonal lengths all lie in the set $\{1, d\}$, where $d > 1$, find all possible values of d.

   Answer: $d = \sqrt{2}$, $\sqrt{3}$, $\dfrac{1+\sqrt{5}}{2}$, $\dfrac{\sqrt{2}+\sqrt{6}}{2}$. (The last value can equivalently be written as $\sqrt{2+\sqrt{3}}$.)

   Solution: There are several cases:

   (a) All four sides of the quadrilateral have length 1, and the two diagonals have length d. In this case, Q is a square, and $d = \sqrt{2}$.

   (b) All four sides of the quadrilateral and one diagonal have length 1, and the other diagonal has length d. In this case, Q is a rhombus obtained by gluing two equilateral triangles together along a side, so $d = \sqrt{3}$.

   (c) Three sides of the quadrilateral have length 1, and one side has length d. In this case, it is not possible for either diagonal to have length 1, so they must both have length d as well. Then the quadrilateral is centrally symmetric, so it is a trapezoid ABCD with base AD. If the diagonals intersect at E, we can calculate that $AE = DE = 1$, and so by similar triangles we observe that $d = \dfrac{1+\sqrt{5}}{2}$. (In fact, this trapezoid is what is obtained by removing one vertex from a regular pentagon.)

(d) The quadrilateral's sides (in order) are $\{1, d, 1, d\}$. In this case, Q is a parallelogram. However, this cannot actually occur, because the long diagonal necessarily has length $> d$.

(e) The quadrilateral's sides (in order) are $\{1, 1, d, d\}$. In this case, Q is a kite: say ABCD where AB = AC = 1, and BD = CD = d, where the diagonals AD and BC intersect at E. If BC = 1, then AE = $\sqrt{3}/2$ and DE = $\sqrt{d^2 - 1/4}$, so since the sum must exceed 1, it would be the case that $\frac{\sqrt{3}}{2} + \sqrt{d^2 - \frac{1}{4}} = d$. However, the square of the left-hand side is $\frac{3}{4} + \sqrt{3}\sqrt{d^2 - \frac{1}{4}} + d^2 - \frac{1}{4} \geq d^2 + \frac{1}{2}$, so it cannot equal d. Thus BC = d, and then AE = $\sqrt{1 - d^2/4} + d\frac{\sqrt{3}}{2}$. The square of this sum must exceed 1, so it is equal to d. Then we obtain $\sqrt{1 - \frac{d^2}{4}} = d\left(1 - \frac{\sqrt{3}}{2}\right)$, so $d^2 = 2 + \sqrt{3} = \frac{4 + 2\sqrt{3}}{2}$.

Taking the square root yields $d = \sqrt{2 + \sqrt{3}} = \frac{\sqrt{2} + \sqrt{6}}{2}$.

(f) Three or four of the quadrilateral's sides have length d. In this case, if A is a vertex with an obtuse angle and B, C are adjacent vertices, then at least one of AB and AC has length d. Then BC (being opposite the obtuse angle A) would thus necessarily have length $> d$, which is impossible.

Remark: If the quadrilateral Q is not required to be convex, the two values $d = \sqrt{3}$ and $d = \frac{\sqrt{2} + \sqrt{6}}{2}$ can also occur in a non-convex quadrilateral, but there are no additional values of d obtained in these cases.

4. The remainder when dividing the polynomial $p(x)$ by $x^3 - 2x^2 - x + 2$ is $ax^2 + 6x + 12$. The remainder when dividing $p(x)$ by $x^3 + x^2 - 4x - 4$ is $3x^2 - bx - 12$. Find $a + b$.

Answer: -3.

Solution: By hypothesis, there exist polynomials $s(x)$ and $t(x)$ such that

$$\begin{aligned} q(x) &= (x^3 - 2x^2 - x + 2)s(x) + (ax^2 + 6x + 12) \\ q(x) &= (x^3 + x^2 - 4x - 4)t(x) + (3x^2 - bx - 12). \end{aligned}$$

Setting $x = -1$ yields

$$\begin{aligned} q(-1) &= 0 \cdot s(-1) + (a + 6) \\ q(-1) &= 0 \cdot t(-1) + (b - 9) \end{aligned}$$

from which we deduce that $a + 6 = 3b - 9$, so that $b = a + 15$. Similarly, setting $x = 2$ yields

$$\begin{aligned} q(-2) &= 0 \cdot s(2) + (4a + 24) \\ q(-2) &= 0 \cdot t(2) + (-2b) \end{aligned}$$

from which we deduce that $4a + 24 = -2b = -2(a + 15)$, from which we obtain $a = -9$ and thus $b = 6$. Then $a + b = \boxed{-3}$.

Remark One such polynomial is $p(x) = -4x^3 - x^2 + 10x + 4$, whose remainder upon dividing by $x^3 + x^2 - 4x - 4$ is $-9x^2 + 6x + 12$ and whose remainder upon dividing by $x^3 + x^2 - 4x - 4$ is $3x^2 - 6x - 12$.

5. The "run-length" of a sequence of heads and tails is the length of the longest consecutive sequence of identical outcomes: thus, the run-length of the sequence HTTHHHHTH is 4, while the run-length of the sequence TTTHHTHHTTT is 3. If a fair coin is flipped 12 times, find the probability that the run-length of the sequence of outcomes is 2.

Answer: 29/256.

Solution: Clearly, there are exactly 2 sequences of run-length 1, as such a sequence must alternate heads and tails. Now we count sequences of run length at most 2. There are four types of such sequences: ones that end in a run of 1H, a run of 2H, a run of 1T, and a run of 2T.

Let a_n be the number of sequences ending in one H, b_n be the number of sequences ending in HH, c_n be the number of sequences ending in T, and d_n be the number of sequences ending in TT. (We implicitly assume that if there is a flip before the given sequence H, HH, T, or TT, then it is of the opposite kind.)

A sequence of length ending in H is obtained by appending an H to a sequence ending in a T. Thus, $a_n = c_{n-1} + d_{n-1}$.
A sequence of length ending in HH is obtained by appending an H to a sequence ending in one H. Thus, $b_n = a_{n-1}$.
A sequence of length ending in T is obtained by appending a T to a sequence ending in an H. Thus, $c_n = a_{n-1} + b_{n-1}$.
A sequence of length ending in TT is obtained by appending a T to a sequence ending in one T. Thus, $d_n = c_{n-1}$.
We can also check easily that $a_1 = c_1 = 1$ and $b_1 = d_1 = 0$. We then recursively calculate

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	1	1	2	3	5	8	13	21	34	55	89	144
b_n	0	1	1	2	3	5	8	13	21	34	55	89
c_n	1	1	2	3	5	8	13	21	34	55	89	144
d_n	0	1	1	2	3	5	8	13	21	34	55	89

and so the total number of such sequences is $2(144 + 89) = 466$. Then there are 464 sequences of run-length 2, so the probability is $\frac{464}{2^{12}} = \boxed{\frac{29}{256}}$.

Remark It is easy to show by induction that $a_n = c_n = F_n$ and $b_n = d_n = F_{n-1}$, where F_n is the nth Fibonacci number. (There are other counting arguments that show this more directly.) Then the total number of possible sequences of run-length 2 of length n is $2F_{n+1} - 2$.

6. Suppose that $1 < a < b < c < d < e$ are positive integers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1$. If e is equal to 3 times a prime number, find the sum of all possible values for d.

Answer: 34. $[(a, b, c, d, e) = (2, 4, 10, 12, 15)$ or $(2, 3, 11, 22, 33).]$

Solution: Suppose $e = 3p$. Imagine rearranging the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1$ to collect all the terms with denominator divisible by p on one side, and the terms with denominator not divisible by p on the other side, and then summing both quantities. Since one side has a denominator not divisible by p, we see that the sum of all terms with denominator divisible by p must (when reduced to lowest terms) have denominator not divisible by p. In particular, since $e = 3p$ is the largest term, the only possibilities for the collection of terms of denominator divisible by p are $\frac{1}{p} + \frac{1}{3p} = \frac{4}{3p}$, $\frac{1}{2p} + \frac{1}{3p} = \frac{5}{6p}$, or $\frac{1}{p} + \frac{1}{2p} + \frac{1}{3p} = \frac{11}{6p}$. We conclude that the only possibilities are $p = 2$, $p = 5$, or $p = 11$, since p must divide the numerator of the sum.

- Observe that $p = 2$ does not work, because $e = 6$ would require $a = 2$, $b = 3$, $c = 4$, $d = 5$, which does not satisfy the equation.
- If $p = 5$, then two terms must be $\frac{1}{10} + \frac{1}{15} = \frac{1}{6}$. The other three terms must have $\frac{1}{f} + \frac{1}{g} + \frac{1}{h} = \frac{5}{6}$ with $f < g < h < 15$. If $f \geq 3$ then $\frac{1}{f} + \frac{1}{g} + \frac{1}{h} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} < \frac{5}{6}$, so we must have $f = 2$. Then $\frac{1}{g} + \frac{1}{h} = \frac{1}{3}$: since $g = 6$ is too large and $g = 5$ does not work, we see that $(g, h) = (4, 12)$ is the only possibility. This gives the unique solution $(a, b, c, d, e) = (2, 4, 10, 12, 15)$, with $d = 12$.
- If $p = 11$, then three terms are $\frac{1}{11} + \frac{1}{22} + \frac{1}{33} = \frac{1}{6}$. The other two terms satisfy $\frac{1}{f} + \frac{1}{g} = \frac{5}{6}$ and it is clear that the only solution is $(f, g) = (2, 3)$. This gives the solution $(a, b, c, d, e) = (2, 3, 11, 22, 33)$, with $d = 22$.

We conclude that the only solutions are $(2, 4, 10, 12, 15)$ and $(2, 3, 11, 22, 33)$, so the sum of all possible d is 34.