Vermont Mathematics Talent Search, Solutions to Test 3, 2014-2015

Test and Solutions by Jean Ohlson and Evan Dummit

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1. At the Mathville Tapas restaurant, the dishes come in three types: small, medium, and large. Each dish costs an integer number of dollars, with the small dishes being the cheapest and the large dishes being the most expensive. (Tax is already included, different sizes have different prices, and the prices have stayed constant for years.) This week, Jean, Evan, and Katie order 9 small dishes, 6 medium dishes, and 8 large dishes. When the bill arrives, the following conversation occurs:
Jean: "The bill is exactly twice as much as last week."
Evan: "The bill is exactly three times as much as last month."
Katie: "If we gave the waiter a 10% tip, the total would still be less than $100."
Find the price of the group's meal next week: 2 small dishes, 9 medium dishes, and 11 large dishes.

   Answer: 97 dollars.

   Solution: Let the prices of the three types of dishes be $\$a$, $\$b$, and $\$c$, where $0 < a < b < c$. By the information given, $9a + 6b + 8c$ is evenly divisible by 2 and by 3: therefore, $\frac{9}{2}a$ is an integer so a is even, and $\frac{8}{3}c$ is an integer, so c is divisible by 3. Furthermore, the total cost of the meal is at most 90 dollars (since 91 dollars plus a 10% tip is more than 100 dollars), so $23a \le 9a + 6b + 8c \le 90$. We conclude that $a < 4$, so since it is even, we must have $a = 2$. Since c is divisible by 3, we either have $c = 6$ or $c \ge 9$, but the latter cannot happen because then $9a + 6b + 8c$ would exceed 90. Thus $c = 6$. Finally, we see that $6b \le 24$, so $b \le 4$. Then $b = 3$ or $b = 4$. However, $b = 4$ is not possible, because then this week's bill would be \$90, meaning that the previous week's bill would have been \$45: this cannot occur if $a = 2$, $b = 4$, and $c = 6$. Thus, $b = 3$, and so next week's meal costs are $2 \cdot 2 + 9 \cdot 3 + 11 \cdot 6 = 97$ dollars.

2. Points A, B, C, and E lie on circle O such that AC is a diameter of O. Point D lies outside the circle on the perpendicular bisector of CE such that angle $DEA = 150°$. If $\angle EAB = 75°$ and $AB = BC = 4$, find the perimeter of pentagon $ABCDE$.

   Answer: $8 + 4\sqrt{2} + 2\sqrt{6}$.

   Solution: Since AC is a diameter of O and B and E lie on O, AEC and ABC are right triangles. Since $AB = BC = 4$, we see $AC = 4\sqrt{2}$, and $\angle CAB = 45°$. Since $\angle EAB = 75°$, we obtain $\angle CAE = 30°$, meaning that $CE = 2\sqrt{2}$ and $AE = 2\sqrt{6}$. Also, since $\angle DEA = 150°$, we see $\angle CED = 60°$, and since D lies on the perpendicular bisector of CE, we have $CD = DE$, so $\triangle CDE$ is equilateral. Then $CD = DE = 2$, so the perimeter of the pentagon $ABCDE$ is $4+4+2\sqrt{6}+2\sqrt{2}+2\sqrt{2} = 8+4\sqrt{2}+2\sqrt{6}$.

3. Let $[x]$ be the greatest integer function and $\{x\} = x - [x]$ be the fractional part of x. For example, $[\pi] = 3$ and $\{\pi\} = \pi - 3$. Find the number of real numbers x in the interval $[0, 1]$ such that $\{x\}+\{2x\}+\{3x\}+\{4x\}+\{5x\} = 2$.

   Answer: 9.

   Solution: Rewrite the given equation as $(x - [x]) + (2x - [2x]) + (3x - [3x]) + (4x - [4x]) + (5x - [5x]) = 2$, which is equivalent to $15x = 2 + [x] + [2x] + [3x] + [4x] + [5x]$. Now observe that the right-hand side is an integer, meaning that $15x$ is an integer. Testing $x = \frac{k}{15}$ for $k = 0, 1, \cdots, 15$ shows that $\{x\} + \{2x\} + \{3x\} + \{4x\} + \{5x\} = 2$ for $k = 2, 3, 4, 5, 6, 8, 9, 10, 12$, and for no other values. Thus, there are 9 such real numbers x.

4. Four boys and four girls each bring one gift to a party. Each boy randomly chooses a girl to give his gift to, and each girl randomly chooses a boy to give her gift to. Determine the probability that each person receives exactly one gift and that no two people exchanged gifts directly with one another (i.e., if B gave G a gift, then G did not give B a gift).

Answer: 27/8192.

Solution: Of the 4^4 ways the boys could distribute their presents, 4! of them will be to different girls, and the same holds for the girls. Thus, the probability that everyone receives exactly one present is $\frac{(4!)^2}{4^8}$. Now we compute the probability that, if everyone receives exactly one present, no two people directly exchanged gifts.

Consider each girl and the boy she gave a present to: label these four pairs A, B, C, and D, and now consider only the presents given by the boys. Each of the pairs gave and received one present from another pair: we want to find the probability that no pair gave itself a present. There could either be two sets of pairs that each gave each other a present (3 ways) or the pairs could pass presents in a circle (6 ways), out of a total of 4! possibilities.

Hence, the total overall probability is $\frac{9}{4!} \cdot \frac{(4!)^2}{4^8} = \frac{27}{2^{13}} = \frac{27}{8192}$.

5. A set of n distinct positive integers has sum 2015. If every integer in the set has the same sum of digits (in base 10), find the largest possible value of n.

Answer: $n = 19$.

Solution: First we show that $n = 19$ is possible. To see this, observe that the collection $\{8, 17, 26, 35, 44, 53, 62, 71, 80, 107, 116, 125, 134, 143, 152, 161, 170, 206, 305\}$ is a set of 19 integers each having digit sum 8, whose collective total is 2015.

Now we show no larger value of n is possible. Let S be the common digit sum. Then each integer is congruent to S modulo 9, so summing all the integers yields $nS \equiv 2015 \equiv 8 \pmod 9$. In particular, since 2015 is not divisible by 3, n cannot be divisible by 3.

Furthermore, all of the integers are congruent modulo 9. Therefore, the kth integer must be at least $9k + 1$, so the sum of all the integers is at least $1 + 10 + 19 + ... + (1 + 9n) = \frac{n(2 + 9n)}{2}$. Since the sum is 2015, we conclude that $\frac{n(2 + 9n)}{2} \leq 2015$, meaning that $n < 22$.

Now since n is less than 22 and not divisible by 3, we must only eliminate the possibility that $n = 20$. In this case, since $nS \equiv 2015 \equiv 8 \pmod 9$, we see that S would necessarily be congruent to 4 modulo 9.

First we try digit sum 4: the smallest 15 such integers are 4, 13, 22, 31, 40, 103, 112, 121, 130, 202, 211, 220, 301, 310, 400, whose sum is already 2220.

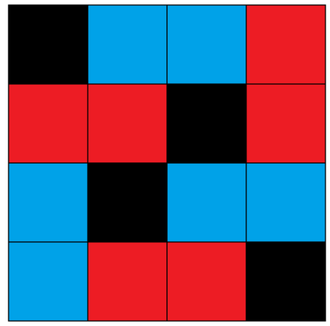
Next we try digit sum 13: the smallest 15 such integers are 49, 58, 67, 76, 85, 94, 139, 148, 157, 166, 175, 184, 193, 229, 238 whose sum is already 2058.

A digit sum of 22 or larger would require every number be at least 400, so the sum would clearly exceed 2015. Thus, $n = 20$ is not possible, so we conclude that the largest possible n is $n = 19$.

Remark: Once $n = 19$ is determined as a possibility, it is easy to see using the congruence relation that the digit sum for $n = 19$ must be congruent to 8 modulo 9. Listing the smallest such integers with digit sum 8 will rapidly lead to the set above.

6. A checkerboard is "almost tileable" if there exists some way of placing non-overlapping dominoes on the board that leaves exactly one square in each row and column uncovered. (Note that dominoes are 2×1 tiles which may be placed in either orientation.) Prove that, for $n \geq 3$, an $n \times n$ checkerboard is almost tileable if and only if n is congruent to 0 or 1 modulo 4.

Solution: We first observe that a 4×4 board is nearly tileable:



Next we observe that if a $4n \times 4n$ board is nearly tileable: divide the board into n^2 4×4 grids. Place n copies of the nearly-tiled 4×4 board on the diagonal, and then fill in each of the other 4×4 grids completely with dominoes.

Now, observe that if $4n \times 4n$ board is nearly tileable, then so is a $(4n+1) \times (4n+1)$ board: tile the lower right $4n \times 4n$ portion as in the smaller board. Then leave the upper-left square uncovered, and pave the strips of length $4n$ on the top and left edges with $2n$ dominoes each.

Now we show that no other boards are almost tileable. Color the board in a standard alternating checkerboard pattern with white and black tiles, with the upper left corner a black tile.

Consider all collections of n squares, no two lying in the same row or column. Define a "move" to be the operation of replacing the squares at (a, b) and (c, d) with the squares at (a, d) and (c, b). By performing a sequence of moves, it is possible to transform any set of n squares into the diagonal of the checkerboard (i.e., the set of squares of the form (i, i)): with the first move, convert the squares $(1, d)$ and $(c, 1)$ into $(1, 1)$ and (c, d). Then convert the squares $(2, d')$ and $(c', 2)$ into $(2, 2)$ and (c', d'). By a trivial induction, repeating the process yields the squares $(1, 1)$, $(2, 2)$, ... , (n, n).

Notice that any move either replaces 2 white squares with 2 black squares, 2 black squares with 2 white squares, or 1 of each with another 1 of each. In each case, the difference $B - W$ of black squares minus white squares remains invariant modulo 4. Hence, for any set of n squares, no two in the same row or column, the difference $B - W$ for the chosen set is equivalent (mod 4) to the difference $B - W$ for the diagonal, which is n.

If $n \equiv 2$ (modulo 4), suppose we have an almost tiling. By the argument above, the difference $B - W$ for the uncovered squares is congruent to $n \equiv 2 \pmod 4$. However, the board contains a total of $\frac{1}{2}n^2$ black squares and $\frac{1}{2}n^2$ white squares, and each domino covers 1 white and 1 black square, so the difference $B - W$ for the uncovered squares must be zero. This is a contradiction, because 0 is not congruent to 2 modulo 4.

If $n \equiv 3$ (modulo 4), again suppose we have an almost tiling. By the argument above, the difference $B - W$ for the uncovered squares is congruent to $n \equiv 3 \pmod 4$. The board contains a total of $\frac{n^2+1}{2}$ black squares and $\frac{n^2-1}{2}$ white squares, and each domino covers 1 white and 1 black square, so the difference $B - W$ for the uncovered squares must be 1. This is a contradiction, because 1 is not congruent to 3 modulo 4.