Vermont Mathematics Talent Search, Solutions to Test 4, 2014-2015

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1. Points A, B, C, and D lie on circle O. Point F lies on CD such that OF is perpendicular to CD, and point E is the intersection of OF with AB. If $AB = 8$, $CD = 6$, $EF = 1$, and OF is perpendicular to AB, find the radius of circle O.

   Answer: 5.

   Solution: Let r be the radius of the circle, and observe that AB and CD are parallel chords. There are two possibilities: either they could be on the same side of O, or they could be on opposite sides of O.

   If they are on opposite sides of O, let $OE = h$: then $OF = 1 - h$. Since $OA = OD = r$, the Pythagorean theorem gives $h^2 + 4^2 = r^2$ and $(1-h)^2 + 3^2 = r^2$. Subtracting the second equation from the first yields $(1-h)^2 - h^2 + 3^2 - 4^2 = 0$, so $-2h + 1 - 7 = 0$ so that $h = -3$. This is impossible.

   We conclude that the chords are on the same side of O. As before, let $OE = h$: then $OF = h + 1$. Since $OA = OD = r$, the Pythagorean theorem gives $h^2 + 4^2 = r^2$ and $(h+1)^2 + 3^2 = r^2$. Subtracting the second equation from the first yields $(h+1)^2 - h^2 + 3^2 - 4^2 = 0$, so $2h + 1 - 7 = 0$ hence $h = 3$, and then $r = 5$.

2. Suppose a, b, and c are integers greater than 1 such that $ab - c < 20$ and $ac - b < 15$. Find the maximum possible value for $bc - a$.

   Answer: 253.

   Solution: The first equation is equivalent to $b < \frac{20+c}{a}$ and the second equation is equivalent to $ac + 15 < b$, so the relations are collectively equivalent to $ac - 15 < b < \frac{20+c}{a}$.

   For a fixed a and c, there will be an integer b satisfying this relation if and only if $ac - 14 < \frac{20+c}{a}$: since $ac - 15$ and b are integers and $ac - 15 < b$, the smallest possible value for b would be $ac - 14$.

   Thus we require $ac - 14 < \frac{20+c}{a}$, which is equivalent to $c < \frac{14a+20}{a^2-1} = \frac{14(a+1)}{a^2-1} + \frac{6}{a^2-1} = \frac{14}{a-1} + \frac{6}{a^2-1}$.

   We observe that for fixed a, because $b < \frac{20+c}{a}$, the largest possible value of c will give the biggest possible upper bound on b, and hence give the largest possible value for $bc - a$. Thus, we only need to compute the largest possible c.

   If $a = 2$, the inequality gives $c < 16$ so the maximum is $c = 15$, giving a maximal $b = 17$ and the maximum value of $bc - a = 253$.

   If $a \geq 3$, then we see that $c < \frac{14}{2} + \frac{6}{8} = \frac{31}{4}$, so the largest c could possibly be is 7. Then $b < \frac{20+c}{a} < \frac{27}{a} \leq 9$, so the maximal b is at most 9. Then $bc - a < 63$, so the maximum does not occur with $a \geq 3$.

   We conclude that the maximum possible value is 253, with $a = 2$, $b = 17$, and $c = 15$.

   Remark: Using $c < \frac{14a+20}{a^2-1}$ we can deduce $b < \frac{20+c}{a} < \frac{14+20a}{a^2-1}$, and then obtain the bound $bc - a < \frac{280 + 595a + 280a^2 + 2a^3 - a^5}{(a^2-1)^2}$. Using calculus, it can be shown that this function is decreasing for $a > 1$, so the largest value of $bc - a$ should occur for small a.

3. Find $\tan^{-1}\left[\sqrt{\frac{2\sin(20°)+\sin(33°)+\sin(7°)}{2\sin(20°)-\sin(33°)-\sin(7°)}}\right]$. Express your answer in simplest form, in degrees.

Answer: $\frac{167}{2}$ degrees.

Solution: For arbitrary x and y, we have

$$\begin{aligned}\frac{\sqrt{2\sin(x)+\sin(x+y)+\sin(x-y)}}{\sqrt{2\sin(x)-\sin(x+y)-\sin(x-y)}} &= \frac{\sqrt{2\sin(x)+2\sin(x)\cos(y)}}{\sqrt{2\sin(x)-2\sin(x)\cos(y)}} \\ &= \frac{\sqrt{2+2\cos(y)}}{\sqrt{2-2\cos(y)}} = \frac{2\left|\cos(y/2)\right|}{2\left|\sin(y/2)\right|} \\ &= \left|\cot(y/2)\right|.\end{aligned}$$

Here, $x = 20°$ and $y = 13°$, so the given expression is $\tan^{-1}\left[\cot(\frac{13°}{2})\right] = 90° - \frac{13°}{2} = \boxed{\frac{167}{2}\text{ degrees}}$.

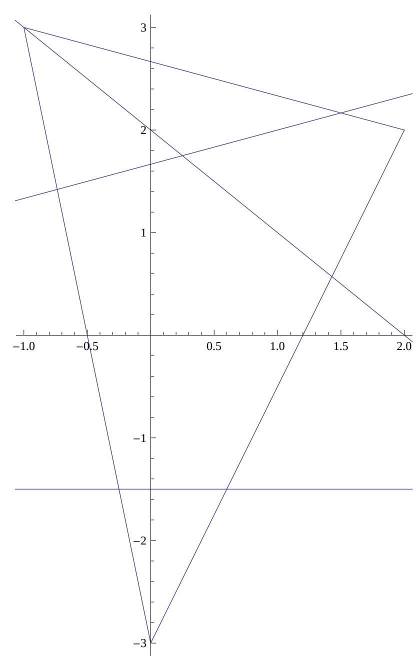
4. A dart is thrown at a triangular dartboard with vertices $(-1, 3)$, $(0, -3)$, and $(2, 2)$, and lands at a random position on the dartboard. Find the probability that the dart is closer to the origin than to any of the three vertices of the dartboard.

Answer: $\frac{49067}{90440}$.

Solution: We use coordinates. The three sides of the triangle have equations $y = -6x - 3$, $y = -\frac{1}{3}x + \frac{8}{3}$, and $y = \frac{5}{2}x - 3$, respectively.

Also, observe that the set of points equidistant from the origin and (a, b) is the perpendicular bisector, which has equation $y = -\frac{a}{b}x + \frac{a^2+b^2}{2b}$.

We see that the three perpendicular bisectors have equations $y = \frac{1}{3}x + \frac{5}{3}$, $y = -\frac{3}{2}$, and $y = -x + 2$. Plotting these lines along with the sides of the triangle shows that the desired region is a pentagon:



Computing the intersections of the appropriate lines yields that the vertices of the pentagon are (in counterclockwise order) equal to $\left(\frac{10}{7}, \frac{4}{7}\right)$, $\left(\frac{1}{4}, \frac{7}{4}\right)$, $\left(-\frac{14}{19}, \frac{27}{19}\right)$, $\left(-\frac{1}{4}, -\frac{3}{2}\right)$, and $\left(\frac{3}{5}, -\frac{3}{2}\right)$.

We then compute the area of this pentagon using the "shoelace formula": this gives $A = \frac{1}{2}c_1 - \frac{1}{2}c_2$, where

$$\begin{aligned} c_1 &= \left[\frac{10}{7}\cdot\frac{7}{4} + \frac{1}{4}\cdot\frac{27}{19} + \left(-\frac{14}{19}\right)\cdot\left(-\frac{3}{2}\right) + \left(-\frac{1}{4}\right)\cdot\left(-\frac{3}{2}\right) + \frac{3}{5}\cdot\frac{4}{7}\right] = \frac{24889}{5320} \\ c_2 &= \left[\frac{1}{4}\cdot\frac{4}{7} + \left(-\frac{14}{19}\right)\cdot\frac{7}{4} + \left(-\frac{1}{4}\right)\cdot\frac{27}{19} + \frac{3}{5}\cdot\left(-\frac{3}{2}\right) + \frac{10}{7}\cdot\left(-\frac{3}{2})\right)\right] = -\frac{1727}{380} \end{aligned}$$

This yields $A = \frac{49067}{10640}$. Since the area of the triangle is $\frac{17}{2}$ (computed in the same way), the desired probability is $\frac{A}{17/2} = \frac{49067}{90440}$.

5. We say that a positive integer k is "3-special" if it has a divisor that a distance of less than 3 from $\sqrt{k}$. For example, 2016 is 3-special: 42 divides 2016, and the distance between 42 and $\sqrt{2016} \approx 44.90$ is less than 3. Prove that there exist seven polynomials $p_1(n)$, $p_2(n)$, ... , $p_7(n)$ such that an integer $k \geq 200$ is 3-special if and only if there exist some integers i and n with $1 \leq i \leq 7$ and $k = p_i(n)$.

Solution: We claim that the desired result holds for the polynomials $p_1(n) = n^2$, $p_2(n) = n(n+1)$, $p_3(n) = n(n+2)$, $p_4(n) = n(n+3)$, $p_5(n) = n(n+4)$, $p_6(n) = n(n+5)$, and $p_7(n) = n(n+6)$.

Observe that for any positive d, we have $\sqrt{n(n+d)} < n + \frac{d}{2}$, so for each d with $0 \leq d \leq 6$, we see that $k = n(n+d)$ is 3-special with special divisor n.

Now suppose k is a 3-special integer with a divisor n within 3 of $\sqrt{k}$. If $n > \sqrt{k}$, then $\frac{k}{n}$ is also a divisor of k, and we claim that it is also within 3 of $\sqrt{k}$: otherwise, we would have $k = n\cdot\frac{k}{n} < (\sqrt{k}+3)(\sqrt{k}-3) = k - 9$, which is impossible.

So (by interchanging n and $\frac{k}{n}$ if necessary) we may assume without loss of generality that $n \leq \sqrt{k}$. Then since $\frac{k}{n}$ is also an integer, we must have $\frac{k}{n} = n + d$ for some integer d, which is to say, $k = n(n+d)$.

It remains to show that if $k \geq 200$, then $k = n(n+d)$ cannot be 3-special with 3-special divisor n for any $d \geq 7$. This would require $n(n+d) < (n+3)^2 = n^2 + 6n + 9$, meaning that $n < \frac{9}{d-6} < 9$, and then $k < (n+3)^2 \leq 121$.

Hence, we conclude that an integer $k \geq 200$ is 3-special if and only if it is of the form $n(n+d)$ for some integer n and some integer d with $0 \leq d \leq 6$.

Remark: In fact, by working through the exceptions to the argument given at the end, it can be shown that the only 3-special numbers not given by any of the seven listed polynomials are 11, 13, 22, 33, 44, 78, and 98.

6. Ten fair six-sided dice are rolled. The probability that no subset of six of these dice have a sum divisible by 6 is $\frac{N}{6^{10}}$. Compute the value of N.

Answer: 1512.

Solution 1: Call a set of 6 dice whose sum is divisible by 6 a "6-set". We start by analyzing subsets of 3 dice whose sum is divisible by 3: call such dice a "3-set".

Suppose we have a set of 10 dice containing no 6-sets, and consider the possible distributions of the rolls into the three residue classes 0 mod 3, 1 mod 3, and 2 mod 3. Observe that three dice form a 3-set if they are all in the same class, or all in different classes.

If there are 7 dice d_1, d_2, ... , d_7 in a single class, then any subset of 6 of them has sum divisible by 3. Since $\{d_1, d_2, \cdots, d_6\}$ is not a 6-set, its sum must be odd. But $\{d_1, \cdots, d_5, d_7\}$ is also not a 6-set, meaning that its sum is also odd: thus, d_6 and d_7 must either be both even or both odd. Repeating this argument shows that all the dice are either even or odd, but this is impossible because then the sum of any six of them would be even, hence any six of them would form a 6-set.

We conclude there are at most 6 dice in any residue class mod 3.

Now we notice that if there are 3 disjoint 3-sets, since each 3-set either has even or odd sum, two of them must be the same, and then this set of 6 dice would have sum divisible by 6.

So now suppose there are two or fewer disjoint 3-sets. There is now a small number of possibilities for the distribution of dice among the residue classes modulo 3:

6-4-0 or 6-3-0: There are two 3-0-0 sets and one 0-3-0 set.

6-2-2: There is one 3-0-0 set and two 1-1-1 sets.

5-4-1: There is one 3-0-0 set, one 0-3-0 set, and one 1-1-1 set.

5-3-2: There is one 3-0-0 set and two 1-1-1 sets.

4-4-2: There is one 3-0-0 set, one 0-3-0 set, and one 1-1-1 set.

4-3-3: There is one 3-0-0 set, one 0-3-0 set, and one 0-0-3 set.

The only remaining possibility is 5-5-0: five dice in one residue class mod 3 and five in another.

Now consider any collection of 3 dice from the first residue class and 3 dice from the second residue class: these dice must have odd sum, else they would be a 6-set. This is true for every subcollection of 3 dice from each residue class, so all 5 dice in the first class must either be even or odd, and the same holds for all 5 dice in the second class. We conclude that the 10 dice must break up into two sets of five identical rolls, where the rolls are different mod 3 and have odd sum.

There are only six ways for this to happen: the sets could be 1-2, 2-3, 3-4, 4-5, 5-6, or 6-1. Hence, the probability is $6 \cdot \binom{10}{5}/6^{10}$, so $N = 6 \cdot \binom{10}{5} = \boxed{1512}$.

Solution 2: Observe that if we add 1 to the result of each die throw (wrapping 6 around to 1), the new set of dice will have the desired property if and only if the old set of dice does. So it is sufficient to break into cases based on the number of times M that the most common die roll occurs, and (by shifting if necessary) we can assume the most common roll is any desired value.

(a) $M \geq 6$: In this case, we clearly have a 6-set.

(b) $M = 5$ or $M = 4$: Assume the most common roll is a 6. No sum of 2 or more other dice can be divisible by 6, so we cannot have more than five 5s, two 4s, one 3, two 2s, or five 1s.

  i. Suppose there is a 2 or 4: then there is only one 2/4, and at most one 5, one 3, and three 1s, so there is at least one 1. Then there is no 5, and there cannot also be both a 3 and three 1s, so there are at most four non-sixes, which is impossible.

  ii. Suppose there is a 3: then there are at most two 5s, two 4s, two 2s, and two 1s. The presence of any die will exclude two of the others (for example, a 5 excludes a 4 and a 1), which is impossible since there are not enough dice left.

  iii. If there is no 2, 3, or 4, then all the dice are 1, 5, or 6. There cannot be both a 1 and a 5, so there are at most five dice that are not 6s. The only possibility is $M = 5$, with five 1s or five 5s, both of which work.

(c) $M = 3$: Again, assume the most common roll is a 6. No sum of 3 or more other dice can be divisible by 6, so we cannot have more than three 5s, two 4s, three 3s, two 2s, or three 1s.

  i. Suppose there is a 3: then there cannot be both a 4 and 5, nor both a 2 and 1, and there can be at most two 5s and at most two 1s. Thus there are at most two dice among the 4s and 5s, and at most two dice among the 1s and 2s. There are at most three 3s and three 6s, meaning that to get ten dice, there must be three 3s, and two pairs among the 1s, 2s, 4s, and 5s. However, this does not work, because with three 3s and three 6s, having any pair of 1, 2, 4, or 5 will exclude the other three.

  ii. Now assume there are no 3s. Consider the pairs {1,4} and {2,5}: since there are seven dice total among these two sets, one set must have at least four dice. Suppose it is {1,4}: since there are at most two 4s and at most three 1s, there are at least two 1s and one 4: but then 6-6-6-4-1-1 is a 6-set. If the set is {2,5}, then by a similar argument, there is a 6-set given by 6-6-6-5-5-2.

(d) $M = 2$: In this case, each of the six outcomes occurs at most twice. If we have 1 of each die roll, then by making 6 the most common die, we see that 1-2-4-5-6-6 is a 6-set. Otherwise, there must be five die rolls that each appear twice. By shifting cyclically, assume that there are no ones: then 2-3-4-4-5-6 is a 6-set.

Thus, the only possibilities with 6 being the most common roll are five 6s and either five 1s or five 5s. Shifting cyclically yields the same results and count as in the first solution.