Vermont Mathematics Talent Search, Solutions to Test 1, 2015-2016

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1. In the following correctly-worked multiplication problem, all of the digits except for the 5s have been erased and some have been replaced with letters. Each letter represents a different digit 0-9 (other than 5), every letter represents the same value everywhere it appears, and none of the integers starts with the digit 0. Find the value of the 8-digit number that is the result of the multiplication problem.

$$
\begin{array}{ccccccccc}
 & & & & & F & I & V & E \\
 & & & \times & & F & I & V & E \\
\hline
 & & & & \_ & \_ & \_ & \_ & F \\
+ & & & \_ & \_ & \_ & \_ & I & \\
+ & & 5 & \_ & 5 & \_ & V & & \\
+ & \_ & \_ & \_ & \_ & E & & & \\
\hline
= & \_ & \_ & 5 & \_ & \_ & \_ & \_ & \_
\end{array}
$$

Answer: $46566976 = 6824^2$.

Solution: From the given letters, we see that the units digit of $E \times E$ is F, the units digit of $E \times V$ is I, the units digit of $E \times I$ is V, and the units digit of $E \times F$ is E. Combining the first and last statements shows that the units digit of $E \times E \times E$ is E, while the units digit of $E \times E$ is F (which is different from E). Checking the 10 possible digits shows that only $E = 4$ and $E = 9$ have both properties. However, if $E = 9$ then $F = 1$, and then $FIVE \times F = FIVE$, which is impossible because the last row of the multiplication dictates that $FIVE \times F$ has 5 digits, not 4. So $E = 4$ and then $F = 6$. Now, since the units digit of $4 \times V$ is I and the units digit of $4 \times I$ is V, both I and V are even. Since they cannot be zero (since they are different) we must have either have $I = 2$ and $V = 8$ or $I = 8$ and $V = 2$. However, the first possibility does not work because the 5s do not end up in the right places, but they do for $I = 8$, $V = 2$. Thus, $FIVE = 6824$ so the answer is $6824^2 = \boxed{46566976}$.

2. A point P is selected at random from the interior of a right triangle with legs 3 and 4. Find the probability that the sum of the squares of the distances from P to the three vertices of the triangle is less than 18.

Answer: $\frac{2\pi}{27}$.

Solution: Embed the triangle in the coordinate plane with vertices at $(0, 0)$, $(3, 0)$, and $(0, 4)$. If P is at (x, y), then the desired sum is equal to $\left[x^2 + y^2\right] + \left[(x-3)^2 + y^2\right] + \left[x^2 + (y-4)^2\right] = 3(x-1)^2 + 3(y-4/3)^2 + \frac{50}{3}$. Therefore, the region where the sum of squares is less than 18 is given by $3(x-1)^2 + 3(y-4/3)^2 + \frac{50}{3} < 18$, or $(x-1)^2 + (y-4/3)^2 < \frac{4}{9}$. This is a circle of radius $\frac{2}{3}$ centered at $(1, \frac{4}{3})$, so it lies completely inside the triangle. Therefore, the desired probability is $\frac{4\pi/9}{6} = \boxed{\frac{2\pi}{27}}$.

Remark: In any triangle (or more generally, in any convex polygon) the point minimizing the sum of squares of the distances to the vertices is the centroid.

3. Find all ordered pairs of integers (x, y) such that

$$\begin{aligned} x^3 &= 89x + 40y \\ y^3 &= 40x + 89y. \end{aligned}$$

Answer: $(x, y) = (0, 0),\ (7, -7), (-7, 7), (8, -5), (-8, 5), (5, -8), (-5, 8)$.

Solution: First, if $y = x$, then both equations reduce to $x^3 = 129x$, which only has the integer solution $(x, y) = (0, 0)$. If $y = -x$ then both equations reduce to $x^3 = 49x$, which gives the additional solutions $(x, y) = (7, -7)$ and $(-7, 7)$.

Now suppose that $y \neq \pm x$. Adding and subtracting the two equations and factoring produces

$$\begin{aligned} (x + y)(x^2 - xy + y^2) &= 129(x + y) \\ (x - y)(x^2 + xy + y^2) &= 49(x - y) \end{aligned}$$

and cancelling the factors of $x + y$ and $x - y$ (which we have assumed to be nonzero) yields

$$\begin{aligned} x^2 - xy + y^2 &= 129 \\ x^2 + xy + y^2 &= 49 \end{aligned}$$

from which we can immediately see that $x^2 + y^2 = 89$ and $xy = -40$. Adding and subtracting twice the second relation from the first gives

$$\begin{aligned} (x + y)^2 &= x^2 + 2xy + y^2 = 9 \\ (x - y)^2 &= x^2 - 2xy + y^2 = 169 \end{aligned}$$

and extracting square roots yields $x + y = \pm 3$ and $x - y = \pm 13$. This gives four additional solutions $(x, y) = (8, -5), (-8, 5), (5, -8)$, and $(-5, 8)$. Thus, there are seven solutions in integers (x, y): $\boxed{(x, y) = (0, 0),\ (7, -7), (-7, 7), (8, -5), (-8, 5), (5, -8), \text{ and } (-5, 8)}$.

Remark: It is possible to find the solutions to the system numerically using a computer or a grapher. However, in order to receive full credit for the problem, it is necessary both to (i) find all 9 solutions to the equation (the seven integer-valued ones along with the solutions $(\sqrt{129}, \sqrt{129})$ and $(-\sqrt{129}, -\sqrt{129})$), and then (ii) show that the system cannot have more than 9 solutions. The second step can be done, for example, by solving the first equation for y and plugging it in to the second equation for x: this yields a degree-9 polynomial equation for x, which by the Fundamental Theorem of Algebra has at most 9 distinct solutions for x. Since y is uniquely determined by x, this shows that the system has at most 9 solutions.

Remark: In general there are nine solutions to the system

$$\begin{aligned} x^3 &= ax + by \\ y^3 &= bx + ay. \end{aligned}$$

It can also be proven that, for a fixed a and b, at most 7 of these solutions can be rational numbers.

4. In triangle ABC, $\tan(A) = 2\tan(B) = 3\tan(C)$. Compute $\cos(A) \cdot \cos(B) \cdot \cos(C)$.

Answer: $1/10$.

Solution: Let $\tan(A) = 6x$: then $\tan(B) = 3x$ and $\tan(C) = 2x$. We know that $C = \pi - A - B$ so $\tan(C) = -\tan(A + B) = -\dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$. This yields $2x = -\dfrac{9x}{1 - 18x^2}$, from which we obtain $36x^3 - 11x = 0$. Since $x > 0$ (otherwise the angles would all be zero or obtuse), we see that $x = \dfrac{\sqrt{11}}{6}$. Then $\tan(A) = \sqrt{11}$, $\tan(B) = \dfrac{\sqrt{11}}{2}$, and $\tan(C) = \dfrac{\sqrt{11}}{3}$, leading to $\cos(A) = \dfrac{1}{\sqrt{12}}$, $\cos(B) = \dfrac{2}{\sqrt{15}}$, and $\cos(C) = \dfrac{3}{\sqrt{20}}$. Then $\cos(A)\cos(B)\cos(C) = \dfrac{6}{\sqrt{3600}} = \boxed{\dfrac{1}{10}}$.

Remark: A faster way to compute x is to use the property that in any triangle ABC, $\tan(A) + \tan(B) + \tan(C) = \tan(A) \cdot \tan(B) \cdot \tan(C)$. This immediately gives $11x = 36x^3$, so that $x = \sqrt{11}/6$.

5. Find all irrational numbers α such that $\alpha^2 - 2\alpha$ and $\alpha^3 - 6\alpha$ are both rational numbers.

Answer: $\alpha = 1 \pm \sqrt{3}$.

Solution 1: Suppose $\alpha^2 - 2\alpha = s$ is rational. We can complete the square to see that $(\alpha-1)^2 = s+1$, meaning that $\alpha = 1 \pm \sqrt{s+1}$. If $\alpha = 1 + \sqrt{s+1}$ we then can expand to get $\alpha^3 - 6\alpha = (s-2)\sqrt{s+1} - 2 + 3s$. Since this quantity and $-2+3s$ are both rational, we see that $(s-2)\sqrt{s+1}$ must also be rational. Since $\sqrt{s+1}$ is irrational (otherwise α would be rational) the only possibility is to have $s - 2 = 0$, giving $\alpha = 1 + \sqrt{3}$. Similarly, if $\alpha = 1 - \sqrt{s+1}$ then we compute $\alpha^3 - 6\alpha = -(s-2)\sqrt{s+1} - 2 + 3s$ so once again the only possibility is $s = 2$ giving $\alpha = 1 - \sqrt{3}$. It is then easy to verify that for $\alpha = 1 \pm \sqrt{3}$ we have $\alpha^2 - 2\alpha = 2$ and $\alpha^3 - 6\alpha = 4$, so these two values do indeed satisfy the requirements.

Solution 2: If $\alpha^2 - 2\alpha = s$ and $\alpha^3 - 6\alpha = t$ are both rational, then α is a root of the polynomials $x^2 - 2x - s$ and $x^3 - 6x - t$, hence also a root of the polynomial $(x^3 - 6x - t) - (x+2)(x^2 - 2x - s) = (s-2)x + (2s-t)$. If $s \neq 2$ then $\alpha = -\dfrac{2s-t}{s-2}$ would be rational, so $s = 2$ and α is a root of the polynomial $x^2 - 2x + 2$, meaning $\alpha = \boxed{1 \pm \sqrt{3}}$. As above, both values are easily seen to work.

Remark: Solution 1 is essentially an explicit reformulation of the more general technique in Solution 2.

6. For fixed positive integers n, a, and b the game of Extreme Hopscotch is played on a line of $n+1$ tiles labeled with the integers 0 through n inclusive. Grace starts at tile 0 and makes a sequence of hops: in each hop, she may either move a tiles forward or b tiles backward (but is not allowed to hop out of the line of tiles). To win the game, Grace must make a sequence of hops starting at tile 0, reaching tile n, and then returning back to tile 0. For example, if $n = 5$, $a = 2$, and $b = 3$, Grace could win via the sequence 0, 2, 4, 1, 3, 5, 2, 4, 1, 3, 0. If a and b are relatively prime positive integers, prove that Grace has a winning strategy if and only if $n \geq a + b - 1$.

Solution: First, we analyze the case $n \leq a + b - 1$. Observe first that, at any position, Grace has at most one legal move: if she is at tile k with $0 \leq k \leq n - a$, then she may only move a tiles forward, and if she is at tile k with $b \leq k \leq n$, she may only move b tiles backward. (These intervals do not overlap because $n - a < b$ by the assumption that $n \leq a + b - 1$.) Thus, we may assume that she makes the only legal move available, if there is one, and take her position to be at tile c_k after k skips. (Thus, $c_0 = 0$, $c_1 = a$, and so forth.) Since there are only finitely many tiles, there must exist i, j with $0 \leq i < j \leq n$ such that $c_i = c_j$; also, because each position can be reached from at most one other position (as there is no position that can be reached with both a forward and backward hop), we see that $c_{i-1} = c_{j-1}$, hence $c_{i-2} = c_{j-2}$, and (by repeating this logic) we see $c_{j-i} = c_0$. In particular, Grace will eventually return to tile 0.

Suppose the first time Grace returns to tile 0 is after t hops, p of which were forward and q of which were backward. Then by the argument above, the positions c_0, c_1, ... , c_{t-1} are all different. Since her position after p forward hops and q backward hops is tile $ap - bq$, we must have $ap = bq$. But a and b are relatively prime, so $a|q$ and $b|p$, hence $p \geq b$ and $q \geq a$, so in particular, we have $t \geq a + b$. But this implies the positions c_0, ... , c_{a+b-1} are all different, which cannot happen if $n < a + b - 1$. Thus Grace cannot win in this case.

If $n = a + b - 1$, however, we see that Grace will win, because the argument above shows that she will necessarily visit all of the squares by taking the only legal move at each step, and end up back at tile 0. (Note in particular that every tile now has exactly one possible move: Grace can hop forward from tiles 0 through $b - 1$, and backward from tiles b through $a + b - 1$.) In particular, she will necessarily visit tile n at some point before returning to tile 0.

Finally, if $n > a + b - 1$, by the argument given above, Grace has a strategy that will allow her to travel from any tile labeled 0 to $a + b - 1$ inclusive to any other such tile. She can then win the n-tile game by using the strategy to travel from 0 to the tile between 0 and $a - 1$ that is congruent to n modulo a, then making forward hops to reach tile n, then making backward skips until she is at a tile between 0 and $b - 1$ again, and finally using the $n = a + b - 1$ strategy to return to tile 0.

Remark: Dr. Grace Hopper was an eminent computer scientist and rear admiral in the United States Navy.