Vermont Mathematics Talent Search, Solutions to Test 2, 2015-2016

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1. Observe that 25 is a perfect square, and the integer obtained by increasing all of its digits by 1 (namely, 36) is also a perfect square. Find the next smallest integer possessing this property: namely, that it is a perfect square, and the integer obtained by increasing all of its digits by 1 is also a perfect square. (Note: it is not possible to increase by 1 the digits of any number containing a digit 9, because $9+1$ is not a base-10 digit.)

   Answer: 2025.

   Solution: Let the smaller integer be $N = a^2$ and the larger integer be b^2. Clearly, N must have at least two digits. If N has exactly 2 digits, then $11 = b^2 - a^2 = (b-a)(b+a)$, so $b = 6$ and $a = 5$, producing $N = 25$. If N has exactly 3 digits, then $10 \le a, b \le 31$ and $111 = b^2 - a^2 = (b-a)(b+a)$, so the only factorization giving acceptable a and b is $a + b = 37$ and $b - a = 3$. However, this produces $b = 20$ and $a = 17$, which do not actually have the required property (since it is not possible to increase all of the digits of $17^2 = 289$ by 1 to obtain $20^2 = 400$). If N has 4 digits, then $32 \le a, b \le 100$ and $11 \cdot 101 = 1111 = b^2 - a^2 = (b-a)(b+a)$, so the only possibility is $b = 56$ and $a = 45$. These integers do work: $45^2 = 2025$ and $56^2 = 3136$, so the answer is $\boxed{2025}$.

   Remark: The next few pairs of squares having this property are $115^2 = 13225$ and $156^2 = 24336$ and $2205^2 = 4862025$ and $2444^2 = 5973136$.

2. In circle O, chords AB and CD intersect at E. If $AE = 3 \cdot BE$ and $CE = 12 \cdot DE$, find $\dfrac{AC}{BD} + \dfrac{BC}{AD}$.

   Answer: 8.

   Solution: Let $DE = x$ so that $CE = 12x$. By power-of-a-point, we have $AE \cdot BE = CE \cdot DE = 12x^2$, so $BE = 2x$ and $AE = 6x$. Then we see immediately that $\triangle AEC$ is similar to $\triangle DEB$ with similarity ratio 2, and $\triangle BEC$ is similar to $\triangle DEA$ with similarity ratio 6, so $\dfrac{AC}{BD} = 2$ and $\dfrac{BC}{AD} = 6$. The desired sum is $\boxed{8}$.

3. A random function f from the set $S = \{1, 2, 3, 4\}$ to the set $T = \{5, 6, 7\}$ is chosen, and a random function g from T to the set $U = \{8, 9\}$ is chosen. Find the probability that the range of the composite function $g \circ f$ is U.

   Answer: $\dfrac{16}{27}$.

   Solution: Suppose that $g \circ f$ has both elements of U in its range: clearly, the range of g must be U, and this occurs with probability $\dfrac{3}{4}$ (since the only functions g whose range is not U are the constant functions). Then g maps two elements x_1 and x_2 of T to the same value in U, and sends the third element x_3 to the other value in U. In order for $g \circ f$ to have both elements of U in its range, f must have x_3 and either x_1 or x_2 in its range. The only way this could fail to happen is if f is the constant function with value x_3 (1 way) or f only takes values of x_1 and x_2 (2^4 ways), so the conditional probability that f has x_3 and either x_1 or x_2 in its range is $1 - \dfrac{1+2^4}{3^4} = \dfrac{64}{81}$.

   Hence, the desired probability is $\dfrac{3}{4} \cdot \dfrac{64}{81} = \boxed{\dfrac{16}{27}}$.

4. In rectangle $ABCD$, there exist points E on BC and F on CD such that triangle AEF is equilateral. If the area of AEF is $16\sqrt{3}$ and the area of $ABCD$ is $34\sqrt{3}$, find the area of triangle CEF.

Answer: $9\sqrt{3}$.

Solution: Observe that $AE = EF = AF = 8$. Now let the measure of angle CEF be x degrees. Then $\angle CFE = 90 - x$, $\angle AFD = 30 + x$, $\angle FAD = 60 - x$, $\angle BAE = x - 30$, and $\angle AEB = 120 - x$, so $AB = 8\cos(x - 30) = 4\sqrt{3}\cos x + 4\sin x$ and $AD = 8\cos(60 - x) = 4\cos x + 4\sqrt{3}\sin x$. Hence the area of $ABCD$ is $16(\sqrt{3}\cos x + \sin x)(\cos x + \sqrt{3}\sin x) = 16\sqrt{3} + 32\sin(2x)$, so $\sin(2x) = \frac{9\sqrt{3}}{16}$. Since $CF = 8\sin(x)$ and $CE = 8\cos(x)$, the area of triangle CEF is $\frac{1}{2}\cdot 8\sin x \cdot 8\cos x = 16\sin(2x)$. Thus, the area of triangle CEF is $\boxed{9\sqrt{3}}$.

5. An "odd Egyptian fraction decomposition into n terms" of a rational number $\frac{p}{q}$ is a representation $\frac{p}{q} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ where $a_1 < a_2 < \cdots < a_n$ are odd positive integers. For example, $\frac{3}{5} = \frac{1}{3} + \frac{1}{5} + \frac{1}{15}$ is an odd Egyptian fraction decomposition of $\frac{3}{5}$ into three terms.

   (a) Prove that there is no odd Egyptian fraction decomposition of 1 into 2016 terms.

   (b) Prove that there is an odd Egyptian fraction decomposition of 1 into 2015 terms.

Solution (part a): Write $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2016}} = \frac{p_1 + p_2 + \cdots + p_{2016}}{a_1 a_2 \cdots a_{2016}}$, where $p_i = (a_1 a_2 \cdots a_{2016})/a_i$. Now each p_i is odd (it is the product of 2015 odd numbers) and the denominator $a_1 a_2 \cdots a_{2016}$ is also odd, so the numerator $p_1 + p_2 + \cdots + p_{2016}$ is a sum of 2016 odd numbers and is therefore even. In particular, the sum has the form $\frac{\text{even}}{\text{odd}}$ and therefore cannot equal 1.

Remark: More generally, the sum of an even number of rational numbers of the form $\frac{n_i}{b_i}$ where n_i and b_i are all odd will have the form $\frac{\text{even}}{\text{odd}}$, and thus cannot equal 1.

Solution (part b): We will show more generally that there is an odd Egyptian fraction decomposition of 1 into n terms for any odd $n \geq 9$. For $n = 9$, we have $1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{21} + \frac{1}{165} + \frac{1}{693}$. Now we show how to transform any decomposition whose last term's denominator is divisible by 3 into a new decomposition with two more terms whose last term's denominator is divisible by 3. To do this, observe that $\frac{1}{3k} = \frac{1}{5k} + \frac{1}{9k} + \frac{1}{45k}$, so we can replace the last term $\frac{1}{3k}$ with $\frac{1}{5k}, \frac{1}{9k}, \frac{1}{45k}$ without changing the total sum or duplicating any terms. Since the last term of the 9-term decomposition we gave is divisible by 3, we may repeat this process to obtain an odd Egyptian fraction decomposition of 1 into 2015 terms.

Remark: There are many different ways to prove part (b) by replacing the last term with a sum of three terms in different ways.

6. Let a and b be real numbers such that $a^4b^3 + a^3b^4 = 2160$ and $a^5b^2 + a^2b^5 = 29520$. Find the value of $(a + 7)(b + 7)$.

Answer: 125.

Solution: Let $s = a + b$ and $p = ab$. Observe that $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$, so $a^3 + b^3 = s^3 - 3sp$. We can then rewrite the equations as

$$\begin{aligned} sp^3 &= 2160 \\ p^2(s^3 - 3sp) &= 29520 \end{aligned}$$

Adding 3 times the first equation to the second equation yields the system

$$\begin{aligned} sp^3 &= 2160 = 2^4 \cdot 3^3 \cdot 5 \\ s^3p^2 &= 36000 = 2^5 \cdot 3^2 \cdot 5^3 \end{aligned}$$

Now taking the cube of the first equation divided by the second equation gives $p^7 = 2^7 3^7$, so $p = 6$, and then $s = 10$. The desired value is $ab + 7(a + b) + 49 = p + 7s + 49 = \boxed{125}$.

Remark: Although it is not necessary to obtain the answer, the conditions $ab = 6$ and $a + b = 10$ yield a quadratic equation in a and b, whose solution is $a, b = 5 \pm \sqrt{19}$.