Vermont Mathematics Talent Search, Solutions to Test 3, 2015-2016

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1. Find all pairs of positive integers (x, y) such that $7x - 4y = 1$ and $y < x\sqrt{3}$.

   Answer: $(3, 5)$, $(7, 12)$, $(11, 19)$.

   Solution 1: Reducing the relation $7x - 4y = 1$ modulo 4 yields $3x \equiv 1 \pmod 4$, so $x \equiv 3 \pmod 4$, meaning that $x = 3 + 4k$ for some $k \geq 0$. From this we immediately see that $y = 5 + 7k$, so we need only find the integers $k \geq 0$ satisfying the relation $(5 + 7k) < (3 + 4k)\sqrt{3}$. Squaring both sides yields $25 + 70k + 49k^2 < 27 + 72k + 48k^2$, or $(k-1)^2 < 3$. This clearly holds only for $k = 0, 1, 2$, so the solutions are $(x, y) = \boxed{(3, 5),\ (7, 12),\ \text{and}\ (11, 19)}$.

   Solution 2: We have $y = \dfrac{7x - 1}{4}$, so plugging into the second equation gives $\dfrac{7x-1}{4} < x\sqrt{3}$. Clearing the denominator and rearranging yields $(7 - 4\sqrt{3})x < 1$. Multiplying both sides by $7 + 4\sqrt{3}$ and observing that $(7 - 4\sqrt{3})(7 + 4\sqrt{3}) = 49 - 48 = 1$ yields $x < 7 + 4\sqrt{3} < 14$, so $1 \leq x \leq 13$. It is then a simple matter to plug in these 13 values of x to see which ones give integral values for y. (Alternatively, we could reduce modulo 4 as in Solution 1 and notice that the desired x will be those congruent to 3 modulo 4.) Either way, we obtain three solutions: $(x, y) = \boxed{(3, 5),\ (7, 12),\ \text{and}\ (11, 19)}$.

2. A (nondegenerate) triangle with side lengths $\cos\theta$, $\cos\theta$, and $2\sin\theta$ has area $\cos 2\theta$. Find the area of the triangle whose side lengths are $\cos^2\theta$, $\cos^2\theta$, and $2\sin^2\theta$.

   Answer: $\sqrt{3}/9$.

   Solution: The original triangle is isosceles with base $2\sin\theta$ and slant height $\cos\theta$, so its height is $\sqrt{\cos^2\theta - \sin^2\theta} = \sqrt{\cos 2\theta}$. Then the triangle's area is equal to $\frac{1}{2}(2\sin\theta)(\sqrt{\cos 2\theta}) = \sin\theta\sqrt{\cos 2\theta}$. By hypothesis this value is equal to $\cos 2\theta$, so since $\cos 2\theta \neq 0$ (otherwise the triangle would be degenerate) we obtain $\sin\theta = \sqrt{\cos 2\theta}$. Squaring both sides yields $\sin^2\theta = 1 - 2\sin^2\theta$, so $\sin^2\theta = \frac{1}{3}$ and then $\cos^2\theta = \frac{2}{3}$. The side lengths of the second triangle are $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}$, so it is equilateral and its area is $\dfrac{(2/3)^2\sqrt{3}}{4} = \boxed{\dfrac{\sqrt{3}}{9}}$.

3. Let circle O have radius 5 with diameter $\overline{AE}$. Point F is outside circle O such that lines $\overline{FA}$ and $\overline{FE}$ intersect circle O at points B and D, respectively. If $FA = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where a, b, c, and d are rational. Find $a + b + c + d$.

   Answer: 15.

   Solution: Observe that $AE = AF$ so triangle AEF is isosceles, and $\angle AEF = \angle AFE = 75°$. Now draw radius $\overline{OB}$: by the central angle property, $m\angle BOE = 60°$, and since $OB = OE = 5$ we see that OBE is equilateral, so $BE = 5$. Since $\angle ABE$ is inscribed in a semicircle, it is right, so $AB^2 + BE^2 = AE^2$ whence $AB = 5\sqrt{3}$.

   Next draw radius $\overline{OD}$: since $OD = OE = 5$ and $\angle OED = 75°$, we see $\angle ODE = 75°$ and $\angle DOE = 30°$. Thus, $\triangle DOE$ is similar to $\triangle FAE$ with similarity ratio $\frac{1}{2}$, so D is the midpoint of FE.

   By power-of-a-point we have $10(10 - 5\sqrt{3}) = FB \cdot FA = FD \cdot FE = 2(FD)^2$, so $FD = DE = \frac{1}{2}(5\sqrt{6} - 5\sqrt{2})$. Furthermore, since $\angle BOD = \angle BOE - \angle DOE = 30°$ we have $BD = DE = \frac{1}{2}(5\sqrt{6} - 5\sqrt{2})$ since they subtend the same angle.

Thus, the perimeter of quadrilateral $ABDE$ is $5\sqrt{3}+2\cdot\frac{1}{2}(5\sqrt{6}-5\sqrt{2})+10=10-5\sqrt{2}+5\sqrt{3}+5\sqrt{6}$, so $a+b+c+d=\boxed{15}$.

4. Right triangle DEF has $E=(2,2)$, while D lies on the curve $y=x^2-x$ and F lies on the curve $y=3x-x^2$. If two vertices of $\triangle DEF$ have the same x-coordinate, find all possibilities for the area of $\triangle DEF$.

Answer: 1, 9, $6\sqrt{2}+8$, $6\sqrt{2}-8$.

Solution: Since E lies on both curves, there are no other points on either curve with the same x-coordinate, so the points D and F must have the same x-coordinate.

If the right angle is not at E, since DF is vertical we see that either DE or EF is horizontal. If DE is horizontal, then D is the other intersection of $y=2$ with $y=x^2-x$, so $D=(-1,2)$ and then $F=(-1,-4)$, for an area of $\frac{1}{2}\cdot 3\cdot 6=9$. If EF is horizontal, then F is the other intersection of $y=2$ with $y=3x-x^2$, so $F=(1,2)$ and then $D=(1,0)$, for an area of $\frac{1}{2}\cdot 1\cdot 2=1$.

If the right angle is at E, then DE and EF are perpendicular. If $D=(a,a^2-a)$ then $F=(a,3a-a^2)$, so DE has slope $\frac{a^2-a-2}{a-2}=1+a$ and EF has slope $\frac{3a-a^2-2}{a-2}=1-a$. These are perpendicular precisely when $(1+a)(1-a)=-1$; namely, when $a=\pm\sqrt{2}$. If $a=\sqrt{2}$, then $DF=4\sqrt{2}-4$ and the distance from E to DF is $2-\sqrt{2}$, so the area of the triangle is $\frac{1}{2}(4\sqrt{2}-4)(2-\sqrt{2})=6\sqrt{2}-8$. If $a=-\sqrt{2}$, then $DF=4\sqrt{2}+4$ and the distance from E to DF is $2+\sqrt{2}$, so the area of the triangle is $\frac{1}{2}(4\sqrt{2}+4)(2+\sqrt{2})=6\sqrt{2}+8$.

Thus, the area is $\boxed{1,\ 9,\ 6\sqrt{2}+8,\ \text{or}\ 6\sqrt{2}-8}$.

5. Find the number of integers n such that the sum of all of the even divisors of n is 2016.

Answer: 14.

Solution: Suppose n has prime factorization $n=2^d p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$. Then the sum of the even divisors of n is equal to $(2+2^2+\cdots+2^d)\cdot\prod_{i=1}^{k}(1+p_i+p_i^2+\cdots+p_i^{a_i})=2\cdot(2^d-1)\prod_{i=1}^{k}\frac{p_i^{a_i+1}-1}{p_i-1}$. After cancelling the factor of 2, we see that each term in the product must therefore be a divisor of $1008=2^4\cdot 3^2\cdot 7^1$. Since 2^d-1 is odd (since $d>0$ as otherwise the product would be zero) and thus divides $3^2\cdot 7$, we see that d must equal 1, 2, 3, or 6.

For p odd, observe that $1+p+p^2>1008$ for $p\geq 32$, and that no term $1+p+p^2$ with p prime divides 1008 for $p\leq 31$. Furthermore, $1+p+p^2+p^3>1008$ for $p\geq 11$, and these terms do not divide 1008 for $p=3$, 5, or 7. It is similarly easy to check that there are no terms of the form $\frac{p^{a+1}-1}{p-1}$ for $p=$3, 5, or 7 and $a>1$ that divide 1008. Hence, no odd prime appears to a power greater than 1 in the factorization of n.

A quick check shows that there are 13 odd primes p such that $p+1$ divides 1008. They, with the factorization of $p+1$, are given in the table:

p	**3**	**5**	**7**	**11**	**13**	**17**	**23**	**41**	**71**	**83**	**167**	**251**	**503**
$p+1$	2^2	$2\cdot 3$	2^3	$2^2\cdot 3$	$2\cdot 7$	$2\cdot 3^2$	$2^3\cdot 3$	$2\cdot 3\cdot 7$	$2^3\cdot 3^2$	$2^2\cdot 3\cdot 7$	$2^3\cdot 3\cdot 7$	$2^2\cdot 3^2\cdot 7$	$2^3\cdot 3^2\cdot 7$

Now we break into cases based on the value of d:

- If $d=6$, then we need $\prod(1+p_i)=2^4$. However, the only terms that can appear in the product are 2^2 and 2^3, and so there is no way to get 2^4.
- If $d=3$, then we need $\prod(1+p_i)=2^4\cdot 3^2$. We can do this with $(23,5)$ or $(17,7)$, yielding $n=920$ and $n=952$.
- If $d=2$, then we need $\prod(1+p_i)=2^4\cdot 3\cdot 7$. We can do this with $(83,3)$, $(41,7)$, $(23,13)$, or $(13,5,3)$, yielding $n=996$, 1148, 1196, and 780.

- If $d = 1$, then we need $\prod(1 + p_i) = 2^4 \cdot 3^2 \cdot 7$. We can do this with $(251, 3)$, $(167, 5)$, $(83, 11)$, $(71, 13)$, $(41, 23)$, $(41, 5, 3)$, $(17, 13, 3)$, or $(13, 11, 5)$, yielding $n = 1506$, 1670, 1826, 1846, 1886, 1230, 1326, and 1430.

In total, there are $\boxed{14}$ such integers n, namely: 780, 920, 952, 996, 1148, 1196, 1230, 1326, 1430, 1506, 1670, 1826, 1846, and 1886.

Remark: It turns out that for the even integers $1 \leq n \leq 2016$, the most common sum of even divisors is 1440, which appears 15 times. The second most common sum is 2016, which (as we saw above) appears 14 times.

6. Suppose a, b, and c are positive real numbers such that $\max(a, b, c) \leq 4 \min(a, b, c)$. Prove that $2ab + 2ac + 2bc \geq a^2 + b^2 + c^2$, and determine when equality can occur.

Answer: Equality can occur precisely when $(a, b, c) = (t, t, 4t)$, $(t, 4t, t)$, or $(4t, t, t)$ for some $t > 0$.

Solution 1: Since everything in the problem is invariant under scaling and reordering the variables, we may assume that $a \leq b \leq c$ and that $a = 1$. Then the problem becomes: if $1 \leq b \leq c \leq 4$, show that $2b + 2c + 2bc \geq 1 + b^2 + c^2$. Completing the square on the right gives $2b + 2c \geq 1 + (c - b)^2$, and we may rearrange this further to give $4b \geq (c - b - 1)^2$. Now since $b \geq 1$ and $-1 \leq c - b - 1 \leq 2$, we see that $4b \geq 4$ whereas $(c - b - 1)^2 \leq 4$. So the desired inequality $4b \geq (c - b - 1)^2$ is always true, and equality can only hold when both terms are equal to 4: that is, when $b = 1$ and $b - c - 1 = 2$, or equivalently, when $(b, c) = (1, 4)$. Therefore, equality in the original problem can hold if and only if $(a, b, c) = (t, t, 4t)$, $(t, 4t, t)$, or $(4t, t, t)$ for some $t > 0$.

Solution 2: As in Solution 1, we may reorder the variables and rescale to assume that $a = 1$ and that $1 \leq b, c \leq 4$. We wish to show that the maximum value of the function $f(b, c) = 2b + 2c + 2bc - 1 - b^2 - c^2$ for $1 \leq b, c \leq 4$ is equal to zero. If we think of this function as a function of b only, with $f(b, c) = -b^2 + (2c + 2)b + (2c - 1 - c^2)$ is a parabola opening downward, so its maximum value on the interval $[1, 4]$ either occurs at an endpoint or at the vertex located at $b = c + 1$. Similarly, as a function of c, the maximum value of $h(c) = -c^2 + (2b + 2)c + (2b - 1 - b^2)$ either occurs at $c = 1$, $c = 4$, or $c = b + 1$. There are nine possible configurations: $(b, c) = (1, 1)$, $(1, 4)$, $(1, b + 1)$, $(4, 1)$, $(4, 4)$, $(4, b + 1)$, $(c + 1, 1)$, $(c + 1, 4)$, $(c + 1, b + 1)$. These respectively give the possible points $(b, c) = (1, 1)$, $(1, 4)$, $(1, 2)$, $(4, 1)$, $(4, 4)$, $(4, 5)$ [not allowed], $(2, 1)$, and $(5, 4)$ [not allowed], with the last case being contradictory. Plugging in the six possible points shows that the maximum value of $f(b, c)$ is 0, occurring at $(b, c) = (1, 4)$ and $(4, 1)$. This proves the required inequality and gives the equality cases as in Solution 1.

Solution 3: Observe that $\sqrt{a} + \sqrt{b} \geq 2\sqrt{\min(a, b, c)} = \sqrt{4 \min(a, b, c)} \geq \sqrt{\max(a, b, c)} \geq \sqrt{c}$, so $\sqrt{a} + \sqrt{b} \geq \sqrt{c}$ with equality if and only if $b = a$ and $4a = c$.

Similarly, $\sqrt{a} + \sqrt{c} \geq \sqrt{b}$ with equality if and only if $a = c$ and $4a = b$, and $\sqrt{b} + \sqrt{c} \geq \sqrt{a}$ with equality if and only if $b = c$ and $4b = a$.

Then

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c}) \geq 0$$

since each term is positive. Multiplying out the right-hand side yields, after some algebra,

$$2ab + 2ac + 2bc - a^2 - b^2 - c^2 \geq 0$$

which is equivalent to the desired inequality. Equality can hold only when one of the terms in the product is equal to zero, which happens precisely when $a = b = c/4$, $a = c = b/4$, or $b = c = a/4$.

Remark: It is also possible to solve this problem using calculus. With the notation of Solution 2, the key point is to observe that there are no critical points of the function $f(b, c)$ inside the square $[1, 4] \times [1, 4]$, as both partial derivatives cannot be equal to zero simultaneously: this would require $2 + 2b - 2c = 2 - 2b + 2c = 0$ which has no solutions. Therefore, any minimum or maximum must therefore occur on the boundary of the square, meaning that at least one of b, c must be equal to 1 or 4. This reduces the problem to a one-variable maximization of a quadratic function whose solutions are straightforward to find.