Vermont Mathematics Talent Search, Solutions to Test 4, 2015-2016

Test and Solutions by Jean Ohlson and Evan Dummit

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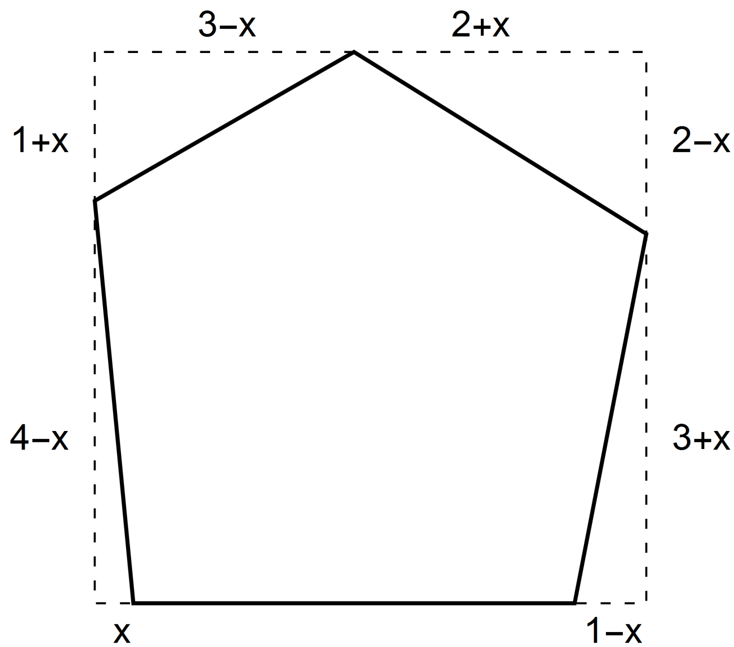
1. Mika is building a garden inside a square plot of land of side length 5 meters. She walks around the perimeter of the plot and places five posts, equally spaced along the 20-meter boundary. Her garden is to be the area inside the pentagon formed by the posts. Find the maximum possible area of her garden, in square meters.

   Answer: 20 square meters.

   Solution: Two posts must be on the same side of the square. Since the posts are 4 meters apart, let one post be x meters from the nearest vertex of the square, where $0 \le x \le 1$. Then Mika's garden appears as follows:



   We can see that the garden consists of the area inside the square with four right triangles removed: the first triangle has legs x and $4 - x$, the second has legs $1 + x$ and $3 - x$, the third has legs $2 - x$ and $2 + x$, and the last has legs $1 - x$ and $3 + x$. Thus, the area of Mika's garden is $25 - \frac{1}{2}x(4 - x) - \frac{1}{2}(1 + x)(3 - x) - \frac{1}{2}(2 - x)(2 + x) - \frac{1}{2}(1 - x)(3 + x) = 2x^2 - 2x + 20 = 2(x - 1/2)^2 + 39/2$. The graph of $y = 2(x-1/2)^2 + 39/2$ therefore has its vertex at $x = 1/2$, meaning that its maximum value for $0 \le x \le 1$ occurs either at $x = 0$ or $x = 1$. Since the value at $x = 0$ and at $x = 1$ are both 20, the maximum is equal to $\boxed{20}$. (Note that the maximum area occurs when one of the vertices of the garden is a vertex of the square.)

2. For their investment club, Alice, Barry, Charlotte, and Duane each buy an integral number of shares of stock from one of four companies, and then sell their shares a year later. Each person only buys shares from one company, and no two people purchased from the same company. The four purchase prices were \$150, \$250, \$300, and \$350, while the four sell prices (not necessarily in the same order) were \$100, \$300, \$400, and \$450. Given that

   (i) Alice bought 6 shares, the most of anyone,

(ii) Barry bought the same number of shares as Charlotte and Duane combined,

(iii) Alice, Charlotte, and Duane each made the same total (positive) profit, and

(iv) Barry invested the second-most money, \$1400, while Charlotte invested the least,

determine the average sell price (per share) of Barry's and Charlotte's combined shares.

Answer: \$170.

Solution: By (iv), Barry's purchase price must divide \$1400, so the only possibility is that his purchase price was \$350, and that he bought 4 shares.

Then Charlotte and Duane together also bought 4 shares. If one of them buys 1 share, then the maximum profit that person could make is $\$450 - \$150 = \$300$, while if they each buy 2 shares, the maximum total profit they could make is $2 \cdot (\$400 + \$450 - \$150 - \$250) = \$900$. By (iii), since Charlotte and Duane make the same total profit, we see that this common profit is at most \$450.

Also by (iii), since Alice made a profit, the difference between her purchase and sell prices is at least \$50, and (by the prices given) must also be divisible by \$50 – so since she bought 6 shares, her profit is a multiple of $6 \cdot \$50 = \$300$. Hence, the profit made by each of Alice, Charlotte, and Duane must be a multiple of \$300. Since it is at most \$450, we conclude that it must be \$300. Then Alice's sell price was \$50 greater than her purchase price, so she must have purchased at \$250 and sold at \$300.

The remaining purchase prices are \$150 and \$300, while the remaining sell prices are \$100, \$400, and \$450. Since Charlotte and Duane each made a profit of \$300, they could not both have bought 2 shares, since there is only one pair of prices that differs by +\$150. Thus, one of them bought 1 share and made a gain of \$300, for which the only possibility is purchase \$150 and sell \$450. Then the other bought 3 shares at \$300 and made a gain of \$100 per share, hence sold at \$400. Since Charlotte invested the least by (iv), she bought 1 share and Duane bought 3. The remaining sell price of \$100 is therefore Barry's. To summarize, we get the following:

Person	# Shares	Purchase	Sell	Net Profit
Alice	6	\$250	\$300	\$300
Barry	4	\$350	\$100	-\$1000
Charlotte	1	\$150	\$450	\$300
Duane	3	\$300	\$400	\$300

Then the total sell price of Barry's and Charlotte's combined 5 shares is \$850, so the answer is $\boxed{\$170}$.

3. Find all bases b such that the result of the base-b arithmetic problem $121_b \cdot 41_b - 215_b \cdot 23_b - 36_b$ is a perfect square. (Note: $b > 6$ because the digit 6 appears in the problem.)

Answer: $b = 12, 14, 28$.

Solution: Upon converting to base 10, the result of the arithmetic is

$$(b^2 + 2b + 1)(4b + 1) - (2b^2 + b + 5)(2b + 3) - (3b + 6) = b^2 - 10b - 20 = (b - 5)^2 - 45.$$

If this is equal to the perfect square n^2 with $n \geq 0$, rearranging and factoring yields

$$(b - 5 - n)(b - 5 + n) = 45.$$

Since $b > 6$, both terms must be positive (because the second one is), so the terms must be $(1, 45)$, $(3, 15)$, or $(5, 9)$. Solving the system yields the pairs $(b, n) = (28, 22)$, $(14, 6)$, and $(12, 2)$. Thus, $b = \boxed{12,\ 14,\ \text{or}\ 28}$.

4. Find all real solutions to the equations

$$\begin{aligned} \sqrt{x+y} + \sqrt{x-y} &= 10 \\ x^2 - y^2 - z^2 &= 476. \\ 2^{\log|y| - \log z} &= 1 \end{aligned}$$

Answer: $(x, y, z) = (26, 10, 10)$ and $(26, -10, 10)$.

Solution: The third equation is equivalent to $\log|y| - \log|z| = 0$, which is in turn equivalent to $|y| = z$. Since this implies $y^2 = z^2$, the first two equations then become

$$\begin{aligned} \sqrt{x+y} + \sqrt{x-y} &= 10 \\ x^2 - 2y^2 &= 476. \end{aligned}$$

Multiplying the first equation by $\sqrt{x+y} - \sqrt{x-y}$ yields $2y = 10(\sqrt{x+y} + \sqrt{x-y})$ so

$$\sqrt{x+y} - \sqrt{x-y} = \frac{1}{5}y.$$

Adding this relation to $\sqrt{x+y} + \sqrt{x-y} = 10$ yields $\sqrt{x+y} = 5 + \frac{1}{10}y$, and then squaring both sides and rearranging gives $x = 25 + \frac{1}{100}y^2$. Plugging this equation for x into $x^2 - 2y^2 = 476$ eventually yields

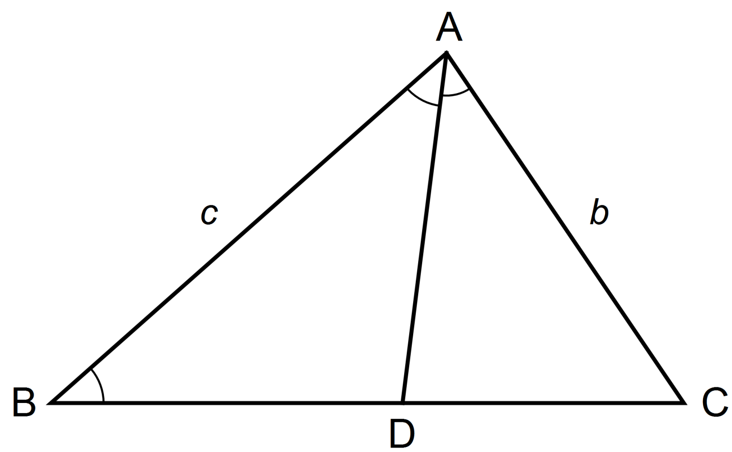
$$\frac{1}{10000}y^4 - \frac{3}{2}y^2 + 149 = 0$$

which factors as $\left(\frac{1}{100}y^2 - 1\right)\left(\frac{1}{100}y^2 - 149\right) = 0$. Thus, $y^2 = 100$ or 14900, from which we obtain four candidate solutions $(x, y, z) = (26, \pm 10, 10)$ and $(174, \pm 10\sqrt{149}, 10\sqrt{149})$. The first pair satisfies the original equations, but the second pair does not: one can calculate that $\sqrt{x+y} + \sqrt{x-y} = 2\sqrt{149}$ rather than 10, because $\sqrt{x \pm y} = \sqrt{149} \pm 5$. Thus we obtain two solutions: $(x, y, z) = \boxed{(26, 10, 10) \text{ and } (26, -10, 10)}$.

5. The side lengths of a triangle are three consecutive positive integers and the largest angle in the triangle is twice the smallest one. Determine the side lengths of the triangle.

Answer: 4, 5, 6.

Solution: Let the triangle be ABC where $AB = c$, $AC = b$, and $BC = a$ with $a > c > b$, and also let the bisector of angle B intersect AC at D with $BD = d$:



Since angle B is twice angle A we see that $\triangle ABD$ is isosceles with $AD = BD$. Furthermore, by the angle bisector theorem, we have $AD = \frac{ac}{b+c} = BD$ and $CD = \frac{ab}{b+c}$. Then Stewart's theorem states $AB^2 \cdot CD + AC^2 \cdot BD = AD^2 \cdot BC + BD \cdot DC \cdot BC$, so plugging in each of the expressions gives

$$c^2 \cdot \left(\frac{ab}{b+c}\right) + b^2 \cdot \left(\frac{ac}{b+c}\right) = \left(\frac{ac}{b+c}\right)^2 \cdot a + \left(\frac{ac}{b+c}\right) \cdot \left(\frac{ab}{b+c}\right) \cdot a.$$

Simplifying and factoring yields

$$abc = \frac{a^3 c}{b+c}$$

which upon cancelling the common factor ac and cross-multiplying gives $b(b+c) = a^2$. Since the sides are consecutive integers we have $a = n$, $b = n-2$, $c = n-1$, whence $(n-2)(2n-3) = n^2$ so $n^2 - 7n + 6 = 0$. This quadratic factors as $(n-1)(n-6) = 0$ so $n = 1$ or $n = 6$. However, $n = 1$ clearly does not work (as b would be negative), so $n = 6$ and the sides of the triangle are $\boxed{4, 5, 6}$.

6. A set of integers is called "3-squarefree" if it contains no 3-element subset the product of whose elements is a perfect square. For example, the set $\{1, 2, 3, 4\}$ is 3-squarefree, since the possible products of a 3-element subset are 6, 8, 12, and 24, none of which is a square. However, the set $\{1, 2, 3, 6\}$ is not 3-squarefree, since the subset $\{2, 3, 6\}$ has product a perfect square.

   (a) Prove that there is no 3-squarefree subset of $\{1, 2, 3, \cdots, 60\}$ containing 45 elements.

   (b) Prove that there is a 3-squarefree subset of $\{1, 2, 3, \cdots, 60\}$ containing 30 elements.

   Solution 1 (part a): Here are 16 disjoint triples of elements each of whose products is a perfect square: $\{1, 13, 52\}$, $\{2, 29, 58\}$, $\{3, 19, 57\}$, $\{4, 10, 40\}$, $\{5, 11, 55\}$, $\{6, 34, 51\}$, $\{7, 28, 36\}$, $\{8, 23, 46\}$, $\{9, 27, 48\}$, $\{12, 33, 44\}$, $\{14, 21, 54\}$, $\{15, 25, 60\}$, $\{16, 20, 45\}$, $\{18, 32, 49\}$, $\{24, 26, 39\}$, $\{30, 35, 42\}$. Any 3-squarefree subset can contain at most 2 of the 3 elements in each triple, so any 3-squarefree subset of $\{1, 2, 3, \cdots, 60\}$ must omit at least 16 elements: thus, it can contain at most 44 elements.

   Solution 2 (part a): Suppose the subset contains no perfect square: then it does not contain any of $\{1, 4, 9, 16, 25, 36, 49\}$. Furthermore, it also can contain at most 2 elements from each of the subsets $\{2, 29, 58\}$, $\{3, 19, 57\}$, $\{5, 11, 55\}$, $\{6, 34, 51\}$, $\{7, 14, 18\}$, $\{8, 23, 46\}$, $\{10, 20, 32\}$, $\{12, 33, 44\}$, $\{15, 40, 27\}$, $\{21, 54, 56\}$, $\{30, 35, 42\}$, $\{45, 48, 60\}$: but then the set contains at most 41 elements. Now suppose the subset contains a perfect square n^2. Then it can contain at most two elements from $\{1, 4, 9, 16, 25, 36, 49\}$, and at most one element from each of $\{2, 8, 18, 32, 50\}$, $\{3, 12, 27, 48\}$, $\{5, 20, 45\}$, $\{6, 24, 54\}$, $\{7, 28\}$, $\{10, 40\}$, $\{11, 44\}$, $\{13, 52\}$, $\{14, 56\}$, $\{15, 60\}$: otherwise, the triple containing n^2 and any two elements in one of these subsets has product a perfect square. However, this leaves at most 38 elements. In either case, the set has at most 41 elements.

   Solution 1 (part b): Say that an integer n has "odd factorization sum" if the sum of the exponents in its prime factorization is odd, and that it has "even factorization sum" if that sum is even: thus, $2 = 2^1$, $12 = 2^2 3^1$, $30 = 2^1 3^1 5^1$, and $32 = 2^5$ have odd factorization sum, while 1, $9 = 3^2$, $15 = 3^1 5^1$, and $60 = 2^2 3^1 5^1$ have even factorization sum. Note that any square has even factorization sum, but the product of any three elements with odd factorization sum also has odd factorization sum. Thus, if we take S to be the set of integers n with odd factorization sum, S is 3-squarefree. Making a list of such integers in the range $[1, 60]$, we see that there are 31 such integers with odd factorization sum: 2, 3, 5, 7, 8, 11, 12, 13, 17, 18, 19, 20, 23, 27, 28, 29, 30, 31, 32, 37, 41, 42, 43, 44, 45, 47, 48, 50, 52, 53, 59. Hence, this set is a 3-squarefree subset of $\{1, 2, 3, \cdots, 60\}$ containing 31 elements.

   Solution 2 (part b): Consider the set of integers of the form $p \cdot 2^a 3^b$ for integers $a, b \geq 0$ and $p \geq 5$ a prime. The product of any three such integers cannot be a square, since it has the form $p_1 p_2 p_3 2^e 3^f$ for primes p_1, p_2, p_3 each at least 5. There are 39 integers in $\{1, 2, 3, \cdots, 60\}$ of this form: 5, 7, 10, 11, 13, 14, 15, 17, 19, 20, 21, 22, 23, 26, 28, 29, 30, 31, 33, 34, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 51, 52, 53, 56, 57, 58, 59, 60. Hence, this set is a 3-squarefree subset of $\{1, 2, 3, \cdots, 60\}$ containing 39 elements.

   Remark: In general, the number of integers with odd factorization sum less than n is very close to $n/2$. (For example, there are 50144 such integers less than 100000.) It was conjectured by Polya that the number of odd-factorization-sum integers less than n is always larger than $n/2$. This turns out not to be true in general, but the first failure only occurs when $n = 906150257$.

   Remark: The best bounds from the solutions show that a maximal 3-squarefree subset of $\{1, 2, 3, \cdots, 60\}$ has at least 39 elements and at most 41: can the argument be sharpened to give an upper bound of 39? Furthermore, what is the general upper bound on the size of a 3-squarefree subset of more general sets like $\{1, 2, 3, \cdots, n\}$? The authors do not know the answers to any of these questions.