Vermont Mathematics Talent Search, Solutions to Test 1, 2016-2017

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1. Jean is given a decimal number with three decimal places. She first rounds it to the nearest hundredth, then to the nearest tenth, then to the nearest integer. (As usual, the digits 0-4 round down and 5-9 round up.) If she finishes with the number 8, what is the positive difference between the largest and smallest possible numbers she could have started with?

   Answer: $0.999 = 999/1000$.

   Solution: First consider the smallest possible number: on the last rounding (up to 8), Jean must have started with at least 7.5, meaning that the previous rounding (up to 7.5) must have started with at least 7.45, and the one before that (up to 7.45) must have started with at least 7.445. For the largest possible number, for the last rounding (down to 8) Jean must have started with at most 8.4, meaning that the previous rounding (down to 8.4) must have started with at most 8.44, and the one before that (down to 8.44) must have started with at most 8.444. Since 7.445 and 8.444 both round to 8 under the procedure given, the desired difference is then $8.444 - 7.445 = \boxed{0.999}$.

2. The 2017 Vermont Mathematics All-Stars A team, consisting of 15 students and 2 coaches, stands in random order in a circle. Find the expected number of students standing between the two coaches (measured the shortest possible way around the circle).

   Answer: $7/2$.

   Solution: Suppose the two coaches are E and J, and label the locations on the circle clockwise starting at E with the integers 0 through 16 inclusive, so that E is labeled with 0. If J is in position k, for $1 \le k \le 16$, then the two arcs between E and J contain $k-1$ and $16-k$ students: if $1 \le k \le 8$ then the first arc is smaller, while if $9 \le k \le 16$ the second arc is smaller. Each of J's possible locations is equally likely, so the expected value is $\frac{1}{16}[0+1+\cdots+7+7+\cdots+1+0] = \frac{1}{16}\cdot 56 = \boxed{\frac{7}{2}}$.

3. Alice and Bob are playing a game: they begin with a pile of k stones, and alternate turns with Alice going first. On each turn, a player may remove n^2 stones for any positive integer n not divisible by 5. Thus, for example, if the pile currently contains 39 stones, a player could remove 1, 4, 9, 16, or 36 of them. The winner of the game is the person who takes the last stone. Determine the number of values of k, $1 \le k \le 2017$, for which Alice has a winning strategy.

   Answer: 1210.

   Motivation: Let us try some small values of k. We tabulate all the possibilities for the number of stones that could remain after one move, starting with a pile of k stones. If any of those moves results in a losing position then k is a winning position (Alice can force Bob to lose), and if all possible moves are to winning positions then k is a losing position (any move Alice makes puts Bob in a winning position):

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Can move to	0	1	2	0,3	1,4	2,5	3,6	4,7	0,5,8	1,6,9	2,7,10	3,8,11	4,9,12	5,10,13	6,11,14
Alice wins?	W	L	W	W	L	W	L	W	W	L	W	L	W	W	L

It appears that if $k \equiv 0, 2 \pmod 5$ then k is a losing position, and if $k \equiv 1, 3, 4 \pmod 5$ then k is a winning position.

Solution: We will show by induction that for any positive integer n, if $n \equiv 0, 2 \pmod 5$ then the position starting with n stones is a losing position, and if $n \equiv 1, 3, 4 \pmod 5$ then the position starting with n stones is a winning position.

For the base case, we analyze the situations with 1, 2, 3, and 4 starting stones. If there are fewer than 4 stones then each player must take 1 at a time, so Alice wins if there are 1 or 3 and Bob wins if there are 2. With 4 stones, Alice clearly can win immediately.

For the inductive step, suppose $k > 4$ is an integer, and assume by hypothesis that every position starting with n stones, for any positive n less than k, is a losing position when $n \equiv 0, 2 \pmod 5$ and a winning position when $n \equiv 1, 3, 4 \pmod 5$. Consider the position starting with k stones. If $k \equiv 1$ or $3 \pmod 5$, then Alice can remove 1 stone and leave Bob in a losing position, since $k - 1 \equiv 0$ or $2 \pmod 5$ is a losing position by hypothesis. If $k \equiv 4 \pmod 5$, then Alice can remove 4 stones and again leave Bob in a losing position with $k - 4 \equiv 0 \pmod 5$ stones. (Note that we are using the assumption that $k > 4$ here.)

On the other hand, if $k \equiv 0$ or $2 \pmod 5$, the number of stones Alice removes must be congruent to 1 or 4 modulo 5, since she may only remove a number that is a square not divisible by 5. Thus, she leaves Bob with a number of stones congruent to one of 1, 3, or 4 modulo 5, which by hypothesis is a winning position for Bob (and thus, a losing position for Alice). This finishes the inductive step, so we are done with the proof.

Returning to the original question, we see that exactly $3/5$ of the integers $k = 1$ through $k = 2015$ inclusive are winning positions, giving 1209 such positions, as is $k = 2017$. This gives a total of $\boxed{1210}$.

4. An *alternating-digit integer* is an integer larger than 9 having the form $ababa\cdots$ for some digits a and b. (We explicitly allow the possibility that $a = b$, that $a = 0$, or that $b = 0$.) The *twin* of the n-digit alternating-digit integer $ababa\cdots$ is the n-digit integer $babab\cdots$. For example, 1313, 111, 30303, and 61616 are alternating-digit integers whose twins are respectively 3131, 111, 03030, and 16161. If P is an alternating-digit integer with an odd number of digits, whose twin Q is less than P, prove that $P^2 - Q^2$ is also an alternating-digit integer.

Solution: Suppose $P = ababab\cdots a$ and $Q = bababa\cdots b$ both have $2n + 1$ digits. Then

$$\begin{aligned} P &= a \cdot (10^{2n} + 10^{2n-2} + \cdots + 10^2 + 1) + b \cdot (10^{2n-1} + 10^{2n-3} + \cdots + 10^3 + 10) \\ &= aR + bS \end{aligned}$$

where we can write $R = \dfrac{10^{2n+2} - 1}{10^2 - 1}$ and $S = \dfrac{10^{2n+1} - 10}{10^2 - 1}$ by summing the geometric series. By symmetry, $Q = bR + aS$, so

$$\begin{aligned} P^2 - Q^2 &= (a^2R^2 + 2abRS + b^2S^2) - (b^2R^2 + 2abRS + a^2S^2) \\ &= (a^2 - b^2)(R^2 - S^2). \end{aligned}$$

We also have

$$\begin{aligned} R^2 - S^2 &= \frac{1}{99^2} \cdot \left[10^{4n+4} - 2 \cdot 10^{2n+2} + 1 - 10^{4n+2} + 2 \cdot 10^{2n+2} - 100\right] \\ &= \frac{1}{99^2} \left[99 \cdot 10^{4n+2} - 99\right] \\ &= \frac{1}{99} \left[10^{4n+2} - 1\right] \\ &= 10^{4n} + 10^{4n-2} + 10^{4n-4} + \cdots + 10^2 + 1. \end{aligned}$$

Since a and b are digits with $b < a$, we see that $1 \le a^2 - b^2 \le 81$ and so $a^2 - b^2$ can be written as a two-digit integer cd (possibly with $c = 0$). Then $(a^2 - b^2)(R^2 - S^2) = cdcdcd\cdots cd$, where there are a total of $2n$ copies of cd. Thus, $P^2 - Q^2$ is an alternating-digit integer, as claimed.

Remark: It is also possible to compute $R^2 - S^2 = 10^{4n} + 10^{4n-2} + \cdots + 10^2 + 1$ directly, by squaring both sums and then cancelling common terms.

5. A circular arc drawn inside a triangle is called a *bisector arc* if it is centered at a vertex of the triangle, the endpoints of the arc lie on the sides of the triangle containing that vertex, and the arc separates the triangle into two pieces of equal area. Triangle ABC has $AB = 2$, $BC = 4$, and $AC = 2\sqrt{3}$. Given that $\triangle ABC$ has three bisector arcs, find (in simplest form) the product of the lengths of the shortest and longest such arcs.

Answer: π.

Solution: Observe that $\triangle ABC$ is a 30-60-90 right triangle with area $\frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = 2\sqrt{3}$. Thus, the area inside the sector made by any bisector arc must be $\sqrt{3}$.

First consider a bisector arc centered at A: since the angle there is 90 degrees, the area of the full circle must be $4\sqrt{3}$. Its radius is therefore $r_A = \sqrt{\frac{4\sqrt{3}}{\pi}} = \frac{2 \cdot 3^{1/4}}{\sqrt{\pi}} \approx 1.485$. Since this radius is less than the altitude to the hypotenuse of $\triangle ABC$, the arc does not extend outside $\triangle ABC$. Thus, there is a bisector arc and its length is $(\pi/2)r_A = \pi^{1/2}3^{1/4} \approx 2.333$.

Next consider a bisector arc centered at B: since the angle there is 60 degrees, the area of the full circle must be $6\sqrt{3}$. Its radius is therefore $r_B = \sqrt{\frac{6\sqrt{3}}{\pi}} = \frac{2^{1/2} \cdot 3^{3/4}}{\sqrt{\pi}} \approx 1.819$. Since this radius is less than the length of AB, the arc does not extend outside $\triangle ABC$. Thus, there is a bisector arc and its length is $(\pi/3)r_B = \pi^{1/2}2^{1/2}3^{-1/4} \approx 1.905$.

Finally consider a bisector arc centered at C: since the angle there is 30 degrees, the area of the full circle must be $12\sqrt{3}$. Its radius is therefore $r_C = \sqrt{\frac{12\sqrt{3}}{\pi}} = \frac{2 \cdot 3^{3/4}}{\sqrt{\pi}} \approx 2.572$. Since this radius is less than the length of AC, the arc does not extend outside $\triangle ABC$. Thus, there is a bisector arc and its length is $(\pi/6)r_C = \pi^{1/2}3^{-1/4} \approx 1.347$.

We see that the longest arc is centered at A and the shortest arc is centered at C. The product of their lengths is $r_A r_C = \pi^{1/2}3^{1/4} \cdot \pi^{1/2}3^{-1/4} = \boxed{\pi}$.

Remark: It can be shown that the bisector arc centered at C is the shortest curve (of any kind) that divides the area of ABC into two equal pieces.

6. Suppose z is a complex number such that $|z+3| = 10$.

   (a) Find the maximum possible value of $\left|z^2+16\right|$.

   (b) Find the minimum possible value of $\left|z^2+16\right|$.

Answers: (a) 185, (b) 60.

Solution 1: Suppose $|z+3| = 10$, so that $z+3 = 10(\cos\theta + i\sin\theta)$ for some angle θ. Then $z = (-3 + 10\cos\theta) + (10\sin\theta)i$ so

$$\begin{aligned} z^2+16 &= \left[16 + (-3+10\cos\theta)^2 - 100\sin^2\theta\right] + 20\sin\theta(-3+10\cos\theta)i \\ &= \left[-75 - 60\cos\theta + 200\cos^2\theta\right] + 20\sin\theta(-3+10\cos\theta)i. \end{aligned}$$

Taking the absolute value and squaring produces

$$\begin{aligned} \left|z^2+16\right|^2 &= \left[-75 - 60\cos\theta + 200\cos^2\theta\right]^2 + 400(1-\cos^2\theta)(-3+10\cos\theta)^2 \\ &= 9225 - 15000\cos\theta + 10000\cos^2\theta \\ &= (100\cos\theta - 75)^2 + 3600 \end{aligned}$$

For part (a), the maximum of $\left|z^2+16\right|^2$ occurs when $\cos\theta = -1$, and the maximum value is $175^2 + 3600 = 34225$. Thus, the maximum value of $\left|z^2+16\right|$ is $\sqrt{34225} = \boxed{185}$.

For part (b), the minimum of $\left|z^2+16\right|^2$ occurs when $\cos\theta = \frac{75}{100} = \frac{3}{4}$, and the minimum value is equal to 3600. Thus, the minimum value of $\left|z^2+16\right|$ is $\sqrt{3600} = \boxed{60}$.

Remark: From the calculations above, we see that the values of z minimizing $\left|z^2+16\right|$ are $z = \frac{9}{2} \pm \frac{5\sqrt{7}}{2}i$, and the value of z maximizing $\left|z^2+16\right|$ is $z = -13$.

Solution 2 (part a): By the triangle inequality, if $|z+3| = 10$ then $|z| \le 13$ with equality holding precisely when $z = -13$. We can then use the triangle inequality again to see that $\left|z^2+16\right| \le |z|^2 + 16 \le 13^2 + 16 = 185$. Equality can hold everywhere when $z = -13$, so the maximum value of $\left|z^2+16\right|$ is 185.

Remark: There does not appear to be a similar argument for the correct lower bound on $\left|z^2+16\right|$. If one tries to mimic the same argument, one would observe $|z| \ge 7$ and then that $\left|z^2+16\right| \ge |z|^2 - 16 \ge 49 - 16 = 33$. However, as shown by Solution 1, this lower bound is not actually attainable.