

Vermont Mathematics Talent Search, Solutions to Test 3, 2016-2017

Test and Solutions by Jean Ohlson and Evan Dummit

January 16, 2017

1. The odd numbers from 5 to 21 inclusive are used to build a 3 by 3 magic square. (In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum.) If 5, 9, and 17 are placed as shown, what is the value of x ?

	5	
9		17
x		

Answer: 11.

Solution: If N is the magic number (the common row, column, and diagonal sum), then summing the three rows yields $3N$. This is also the sum of all the entries in the magic square, which is $5+7+\dots+21 = 117$, so $N = 117/3 = 39$. Thus, the center entry must be $39 - 17 - 9 = 13$, and the entry below it is $39 - 13 - 5 = 21$.

	5	
9	13	17
x	21	

The two unlabeled entries in the top row must have sum 34 and thus be 15 and 19 in some order, and the entries in the bottom row must be 7 and 11. The left column must have the two entries 11 and 19, since they are the only ones which sum to 30. The full square is thus as below, and we see $x = \boxed{11}$.

19	5	15
9	13	17
11	21	7

2. Regular decagon $ABCDEFGHIJ$ is inscribed in circle O , which has radius 2. The point P is located one-third of the way from A to B along the arc AB . Compute the sum of the 10 squared lengths $AP^2 + BP^2 + CP^2 + \dots + JP^2$.

Answer: 80.

Solution: Observe that, since AF , BG , CH , DI , and EJ are all diameters of the circle (since the vertices are opposite one another), that PAF , PBG , PCH , PDI , and PEJ are all right triangles with the right angle at P . Thus, $AP^2 + FP^2 = BP^2 + GP^2 = CP^2 + HP^2 = DP^2 + IP^2 = EP^2 + JP^2 = 4^2$, so the desired sum is $5 \cdot 4^2 = \boxed{80}$.

Remark: As the solution shows, the actual location of the point P on the circle is irrelevant, since the answer is the same no matter where P is. (The given location was intended as a red herring.)

3. Krysta has two fair six-sided dice that have special labels: her blue die is labeled with the six standard trigonometric functions (sine, cosine, tangent, secant, cosecant, cotangent) and her red die is labeled with the six angles 30° , 45° , 60° , 120° , 135° , 150° . Krysta rolls the two dice once, and computes the value x obtained by evaluating the trigonometric function on the blue die at the angle on the red die. She rolls both dice again and obtains the value y in the same manner. Find the probability that $x > y$.

Answer: 101/216.

Solution: There are three possibilities: $x > y$, $x = y$, or $x < y$. By symmetry, the probability that $x > y$ is the same as the probability that $x < y$, so it is sufficient to calculate the probability that $x = y$. We can do this using a table of outcomes:

	30°	45°	60°	120°	135°	150°
sin	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2
cos	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	-1/2	$-\sqrt{2}/2$	$-\sqrt{3}/2$
tan	$1/\sqrt{3}$	1	$\sqrt{3}$	$-\sqrt{3}$	-1	$-1/\sqrt{3}$
cot	$\sqrt{3}$	1	$1/\sqrt{3}$	$-1/\sqrt{3}$	-1	$-\sqrt{3}$
sec	$2/\sqrt{3}$	$\sqrt{2}$	2	-2	$-\sqrt{2}$	$-2/\sqrt{3}$
csc	2	$\sqrt{2}$	$2/\sqrt{3}$	$2/\sqrt{3}$	$\sqrt{2}$	2

There are six entries that appear 3 times each in the table ($1/2, \sqrt{2}/2, \sqrt{3}/2, 2/\sqrt{3}, \sqrt{2}, 2$), six entries that appear 2 times each in the table ($\pm 1/\sqrt{3}, \pm 1, \pm\sqrt{3}$), and six entries that appear once in the table ($-1/2, -\sqrt{2}/2, -\sqrt{3}/2, -2, -\sqrt{2}, -2/\sqrt{3}$). Thus, the probability that $x = y$ is $\frac{18}{36} \cdot \frac{3}{36} + \frac{12}{36} \cdot \frac{2}{36} + \frac{6}{36} \cdot \frac{1}{36} = \frac{7}{108}$,

and so the probability that $x > y$ is $\frac{1}{2} \left[1 - \frac{7}{108} \right] = \frac{101}{216}$.

4. Cyclic quadrilateral $ABCD$ has distinct integer side lengths and a circle O inscribed in it. Find the minimal possible integer value for the area of $ABCD$.

Answer: 12.

Solution: If the sides of $ABCD$ are a, b, c , and d , then since it has a circle inscribed in it, we see that $a + c = b + d$. Furthermore, the area is given by Brahmagupta's formula as $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{1}{2}(a+b+c+d) = a+c = b+d$, so the area simplifies as $K = \sqrt{c \cdot d \cdot a \cdot b} = \sqrt{abcd}$. Thus, we are seeking four distinct integers a, b, c, d such that $a + c = b + d$ and $abcd$ is a perfect square. To find such integers, note that K cannot be a prime power p^n : otherwise, if $a = p^k$ and b, c, d are larger powers of p , taking $a + c = b + d$ modulo p^{k+1} yields a contradiction. Also, K cannot be the product of two primes pq : the only possibility with $abcd = p^2q^2$ that avoids duplicate factors is for a, b, c, d to be $1, p, pq, q$ (or a permutation), but then since p and q are both greater than 1, the only choice for opposite side pairs is to have 1 paired with pq . However, $1 + pq = p + q$ is equivalent to $(p-1)(q-1) = 1$, which has only the solution $p = q = 2$, which does not work. The smallest value of K that is not a prime power or product of two primes is $K = 12$, and this works: $12^2 = 1 \cdot 2 \cdot 8 \cdot 9$ and $1 + 9 = 2 + 8$. Therefore, 12 is the smallest possible integer area.

Remark: An infinite family of such quadrilaterals with integral area is given by $(a, b, c, d) = (1, 1 + t + t^2, t^3, t + t^2 + t^3)$ for positive integers t . (The reader is invited to find other such families!)

5. Let a, b , and c be the three positive real roots of the cubic polynomial $p(x) = x^3 - 7x^2 + 11x - 4$, and set $r = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find the unique ordered triple of integers (s, t, u) with the property that $r^4 + sr^3 + tr^2 + ur$ is an integer.

Answer: $(s, t, u) = (0, -14, -16)$.

Solution: Since $p(x) = (x-a)(x-b)(x-c)$, equating coefficients yields $a + b + c = 7$, $ab + ac + bc = 11$, and $abc = 4$. Now,

$$r^2 = a + b + c + 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$$

so

$$r^2 - 7 = 2\sqrt{ab} + 2\sqrt{ac} + 2\sqrt{bc}.$$

Squaring again yields

$$\begin{aligned} (r^2 - 7)^2 &= 4ab + 4ac + 4bc + 8\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \\ &= 44 + 16r. \end{aligned}$$

Thus, r satisfies the equation $(r^2 - 7)^2 = 44 + 16r$, so $r^4 - 14r^2 - 16r = -5$. We may therefore take $(s, t, u) = \boxed{(0, -14, -16)}$.

Remark: It was not expected for students to consider whether the triple (s, t, u) is unique (the wording directly indicates it is), but we will briefly justify uniqueness here. We have shown above that r is the root of a degree-4 polynomial $q(x) = x^4 - 14x^2 - 16x + 5$ with integer coefficients: if we can show that r is not the root of any other monic polynomial of degree ≤ 4 , that would suffice to show the uniqueness of (s, t, u) . Let $m(x)$ be the monic polynomial with integer coefficients of minimal degree that has r as a root. Consider the polynomial greatest common divisor of $m(x)$ and $q(x)$: it also has integer coefficients and has r as a root. By definition, the gcd divides $m(x)$, but since $m(x)$ was assumed to have minimal possible degree, the gcd must therefore be exactly $m(x)$. We conclude that $m(x)$ divides $q(x)$. But now it is a straightforward (if tedious) verification that $q(x)$ does not have any nontrivial factorization into polynomials with integer coefficients. Thus, $m(x)$ must equal $q(x)$, and we are done.

6. For a positive integer n , let $p(n)$ denote the product of the positive divisors of n : thus, $p(6) = 36$. Recursively define $p^d(n) = p(p^{d-1}(n))$ for each $d \geq 2$, with $p^1(n) = p(n)$: thus, $p^2(6) = 10077696$. Prove that $\frac{1}{2} \cdot 3^{2016} < \log_2(\log_6(p^{2017}(6))) < 3^{2016}$.

Solution: Let $d(n)$ denote the number of divisors of n . We begin by observing that $p(n) = n^{d(n)/2}$: if we instead consider $p(n)^2$, then we can group the terms of the product into pairs $(r, n/r)$. The product of each pair is n and there are $d(n)$ such pairs, so $p(n)^2 = n^{d(n)}$ and thus $p(n) = n^{d(n)/2}$. Using this formula, we can compute that $p(6^k) = (6^k)^{(k+1)^2/2} = 6^{k(k+1)^2/2}$, since $6^k = 2^k 3^k$ has $(k+1)^2$ divisors. From here we can immediately see that $p^d(6)$ will always be a power of 6: specifically, $p^d(6) = 6^{a_d}$, where $a_1 = 2$ and $a_d = \frac{1}{2}a_{d-1}(a_{d-1} + 1)^2$ for $d \geq 2$. For example, we can compute $a_2 = 9$, $a_3 = 450$, and $a_4 = 45765225$.

If we make the simple observation that $a_d > \frac{1}{2}a_{d-1}^3$, we see that $a_2 > 2^2$, $a_3 > 2^5$, $a_4 > 2^{14}$, ... , and by an easy induction, $a_d > 2^{(3^{d-1}+1)/2}$. Thus in particular, $a_{2017} > 2^{(3^{2016}+1)/2} > 2^{3^{2016}/2}$.

We can also observe that $a_d < a_{d-1}^3$ for $d \geq 3$ since $(a_{d-1} + 1)^2/2 < a_{d-1}^2$ whenever $a_d \geq 3$, so we see that $a_3 < 2^9$, $a_4 < 2^{27}$, and by an easy induction, $a_d < 2^{3^{d-1}}$. Thus in particular, $a_{2017} < 2^{3^{2016}}$.

Since $a_{2017} = \log_6(p^{2017}(6))$, combining the two inequalities yields $\frac{1}{2} \cdot 3^{2016} < \log_2(\log_6(p^{2017}(6))) < 3^{2016}$, as required.

Remark: Using the same arguments, we can substantially strengthen both the upper bound and the lower bound. Let $b_d = \log_2 a_d$.

For the lower bound, the inequality $a_d > \frac{1}{2}a_{d-1}^3$ implies $b_d - \frac{1}{2} > 3(b_{d-1} - \frac{1}{2})$, so iterating this relation yields $b_{2017} > \frac{1}{2} + 3^{2017-k}(b_k - 1/2) > 3^{2016} \cdot \frac{b_k - 1/2}{3^{k-1}}$ for any $k \geq 1$. Taking $k = 4$, for instance, yields $b_{2017} > 3^{2016} \cdot \frac{\log_2(45765225) - 1/2}{3^3} > 3^{2016} \cdot 0.923990$.

For the upper bound, the inequality $a_d < 0.5001a_{d-1}^3$ holds for $d \geq 5$ because $n(n+1)^2/2 < 0.5001n^3$ whenever $n \geq 10001$. Letting $r = -\frac{1}{2}\log_2(0.5001) \approx 0.49986$, we obtain $(b_d - r) < 3(b_{d-1} - r)$. Iterating this relation yields $b_{2017} < r + 3^{2017-k}(b_k - r) < r + 3^{2016} \cdot \frac{b_k - r}{3^{k-1}}$. Taking $k = 4$, for instance, yields $b_{2017} < r + 3^{2016} \cdot \frac{\log_2(45765225) + \log(0.5001)/2}{3^3} < 3^{2016} \cdot 0.923997$.

Thus, we have the stronger inequalities $0.923990 \cdot 3^{2016} < \log_2(\log_6(p^{2017}(6))) < 0.923997 \cdot 3^{2016}$.