Vermont Mathematics Talent Search, Solutions to Test 4, 2016-2017

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1. Let ABCDEFGH be a regular octagon of side length 4. Construct squares ABIJ and BCKL so that I, J, K, and L all lie inside the octagon. Find the area of the region of overlap between squares ABIJ and BCKL.

   Answer: $16(\sqrt{2}-1)$.

   Solution: The angle ABC is 135 degrees, and each of angles CBL and ABI is 90 degrees, hence angle IBL is 45 degrees. Denote the point of concurrency of the bisector of this angle and the sides IJ and KL by X. Then angle IBX has measure 22.5 degrees, and the area of the kite-shaped region of overlap is just the product of the lengths BI and IX. Since $\dfrac{IX}{BI} = \tan(\dfrac{45^\circ}{2}) = \dfrac{1-\cos(45^\circ)}{\sin(45^\circ)} = \dfrac{1-\sqrt{2}/2}{\sqrt{2}/2} = \sqrt{2}-1$ and $BI = 4$, it follows that the area of the region of overlap is $4^2(\sqrt{2}-1) = \boxed{16(\sqrt{2}-1)}$.

2. If ABC is a nondegenerate triangle and $(1-2\tan(A))(1-2\tan(B)) = 5$, find all possible values of $\sin(C)$.

   Answer: $\sin(C) = \dfrac{2\sqrt{5}}{5}$ (only one possible value).

   Solution: Expanding out the given equation yields $1-2\tan(A)-2\tan(B)+4\tan(A)\tan(B) = 5$, which can be rearranged to $\tan(A)+\tan(B) = 2\tan(A)\tan(B)-2$, which in turn would imply $\dfrac{\tan(A)+\tan(B)}{\tan(A)\tan(B)-1} = 2$ provided that $\tan(A)\tan(B) \neq 1$. If it were the case that $\tan(A)\tan(B)$ were 1, then we would get $\tan(A)+\tan(B) = 0$, but this is impossible since it would force $\tan^2(A) = -1$. So $\dfrac{\tan(A)+\tan(B)}{\tan(A)\tan(B)-1} = 2$. Now, by tangent identities, we have $\tan(C) = \tan(\pi - A - B) = -\tan(A+B) = \dfrac{\tan(A)+\tan(B)}{\tan(A)\tan(B)-1} = 2$, so $\tan(C) = 2$. The Pythagorean identities then yield $\dfrac{1}{\sin^2(C)} = \csc^2(C) = 1+\cot^2(C) = 5/4$, so since $\sin(C)$ is positive (as C is an angle of a triangle) we see the only possibility is $\sin(C) = \boxed{\dfrac{2\sqrt{5}}{5}}$.

   Remark: There exist infinitely many such triangles: specifying any value for $\tan(A)$ other than 1/2 uniquely determines a triangle satisfying the conditions.

3. For any $n \geq 3$, an n-digit integer is called a "canyon integer" if there is an integer k, $2 \leq k \leq n-1$, such that its first k digits form a strictly decreasing sequence, and its last $n-k+1$ digits form a strictly increasing sequence. For example, 543212345, 976124, and 302 are canyon integers, whereas 987, 55234, 82196, and 1358 are not canyon integers. Find the total number of canyon integers. [Note: integers do not start with the digit 0.]

   Answer: 347489.

   Solution: We tabulate the canyon integers depending on the value of their smallest digit (which is the kth digit, in the definition given): if the smallest digit is d, then the set of digits appearing before d is a nonempty subset A of $\{d+1,\dots,9\}$ arranged in decreasing order. Likewise, the set of digits appearing after d is a nonempty subset B of $\{d+1,\dots,9\}$ arranged in increasing order. The canyon integers with smallest digit d are therefore in bijection with pairs (A,B) of nonempty subsets of $\{d+1,\dots,9\}$, so there are $(2^{9-d}-1)^2 = 2^{2(9-d)} - 2^{10-d} + 1$ such canyon integers. Summing over all possible values of d, we see that in total, there are

$$\begin{aligned}
\sum_{d=0}^{9} \left[2^{2(9-d)} - 2^{10-d} + 1\right] &= \left[2^0 + 2^2 + \cdots + 2^{18}\right] - \left[2^1 + 2^2 + \cdots + 2^{10}\right] + 10 \\
&= \frac{2^{20}-1}{2+1} - (2^{11}-2) + 10 \\
&= \boxed{347489}
\end{aligned}$$

canyon integers.

4. Find all integers n for which $n^3 - 10n^2 + 20n + 17$ is the cube of an integer.

Answer: $n = -11,\ 4,\ 5$.

Solution: Let $p(n) = n^3 - 10n^2 + 20n + 17$. We claim that unless $-11 \le n < 10$, we have the inequalities $(n-4)^3 < p(n) < (n-3)^3$, from which we immediately see that $p(n)$ cannot be the cube of an integer. To show the result, first note that $p(n) - (n-4)^3 = 2n^2 - 28n + 81 = 2(n-7)^2 - 17$, so unless $|n-7| < 3$, which is to say, $4 < n < 10$, we have $p(n) > (n-4)^3$. In a similar way, $(n-3)^3 - p(n) = n^2 + 7n - 44 = (n+7/2)^2 - 225/4$, so unless $|n+7/2| \le 15/2$, which is to say $-11 \le n \le 4$, we have $p(n) < (n-3)^3$. Putting the above inequalities together, we see that unless $-11 \le n < 10$, it is true that $(n-4)^3 < p(n) < (n-3)^3$. Thus, any value of n for which $p(n)$ is the cube of an integer must lie in this range. Evaluating $p(n)$ for $-11 \le n \le 10$ yields the following:

n	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$p(n)$	-2744	-2183	-1702	-1295	-956	-679	-458	-287	-160	-71	-14

n	0	1	2	3	4	5	6	7	8	9	10
$p(n)$	17	28	25	14	1	-8	-7	10	49	116	217

We see that the values of n for which $p(n)$ are a perfect cube are $n = \boxed{-11,\ 4,\ 5}$.

5. Find the minimum value of $f(a,b) = [3\cos(a) - 4\cos(b) + 12]^2 + [3\sin(a) - 4\sin(b) + 5]^2$.

Answer: 36.

Solution 1: Observe that we can parametrize the points on the circle $x^2 + y^2 = 3^2$ as $(3\cos(a), 3\sin(a))$ for $0 \le a \le 2\pi$ and the points on the circle $(x+12)^2 + (y+5)^2 = 4^2$ as $(4\cos(b) - 12, 4\sin(b) - 5)$ for $0 \le b \le 2\pi$. The distance between a point on the first and second circles is then equal to the square root of the quantity $[3\cos(a) - (4\cos(b) - 12)]^2 + [3\sin(a) - (4\sin(b) - 5)]^2$, which is precisely $f(a,b)$. We can therefore reinterpret the problem as asking us to compute the square of the minimal distance between a point on $x^2 + y^2 = 3^2$ and a point on $(x+12)^2 + (y+5)^2 = 4^2$. The distance between the centers of these circles is $\sqrt{12^2 + 5^2} = 13$, so the minimal distance is $13 - 3 - 4 = 6$, obtained by the two points lying on the line segment joining the centers of the circles. Thus, the minimum of the given function is $6^2 = \boxed{36}$.

Solution 2: Expanding the given expression yields

$$f(a,b) = 194 - 24\cos(a-b) + 30\sin a + 72\cos a - 40\sin b - 96\cos b.$$

Now set $x = a + \arccos(12/13)$ and $y = b + \arccos(12/13)$, so that so that $\cos(x) = \frac{12}{13}\cos(a) + \frac{5}{13}\sin(a)$, $\cos(y) = \frac{12}{13}\cos(b) + \frac{5}{13}\sin(b)$, and $\cos(a-b) = \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$. Thus, we see that

$$f(a,b) = 194 - 24\cos(x)\cos(y) - 24\sin(x)\sin(y) + 78\cos(x) - 104\cos(y)$$

so it suffices to minimize this function of x and y instead. Observe that when $x = \pi$ and $y = 0$, we have $f(a,b) = 36$, so it is enough to prove that $f(a,b) \ge 36$. Also notice that $p\cos(x) + q\sin(x) = \sqrt{p^2+q^2}\cos(x+\alpha)$ where $\alpha = \arccos(p/\sqrt{p^2+q^2})$, so $p\cos(x) + q\sin(x) \ge -\sqrt{p^2+q^2}$. Thus, we can write

$$\begin{aligned}
f(a,b) &= 194 + [78 - 24\cos(y)]\cos(x) + [24\sin(y)]\sin(x) - 104\cos(y) \\
&\ge 194 + \sqrt{(78 - 24\cos(y)^2 + (24\sin(y))^2} - 104\cos(y) \\
&= (3 + \sqrt{185 - 104\cos(y)})^2
\end{aligned}$$

and since $3 \le \sqrt{185 - 104\cos(y)} \le 17$, we conclude that $f(a, b) \ge 6^2 = 36$.

Remark: Working through the equality case from either proof shows that the minimum occurs precisely when $(\cos a, \sin a) = (-12/13, -5/13)$ and $(\cos b, \sin b) = (12/13, 5/13)$.

6. Let $p(t) = 2t^4 - t^3 - 4t^2 + t + 1$, and let the four real roots of $p(t)$ be $a_1 < a_2 < a_3 < a_4$. Also let $q(t) = 4t^4 - 41t^3 + t^2 + 364t + 181$, and let the four real roots of $q(t)$ be $b_1 < b_2 < b_3 < b_4$. Prove that the four points $(x, y) = (a_1, b_1)$, (a_2, b_2), (a_3, b_3), and (a_4, b_4) all lie on the parabola $y = x^2 + 4x + 1$.

Solution 1: We want to show that the four points lie on $y + 3 = (x + 2)^2$. Equivalently, if we define

$$f(t) = p(t - 2) = 2t^4 - 17t^3 + 50t^2 - 59t + 23$$

and

$$g(t) = q(t - 3) = 4t^4 - 89t^3 + 586t^2 - 1181t + 529$$

then we wish to show that the four roots $c_1 < c_2 < c_3 < c_4$ of $f(t)$, where $c_i = a_i + 2$, and the four roots $d_1 < d_2 < d_3 < d_4$ of $g(t)$, where $d_i = b_i + 3$, have the property that $c_i^2 = d_i$ for $i = 1, 2, 3, 4$. For this, it is sufficient to prove that the squares of the roots of $f(t)$ are the roots of $g(t)$, because all of the roots of f are positive real numbers. (This follows by observing that the values of f alternate in sign when evaluated at $t = 0, 1, 2, 3, 4$.)

For this, factor f as $f(t) = 2(t - c_1)(t - c_2)(t - c_3)(t - c_4)$. Then

$$\begin{aligned} f(\sqrt{t})f(-\sqrt{t}) &= 4(\sqrt{t} - c_1)(-\sqrt{t} - c_1)(\sqrt{t} - c_2)(-\sqrt{t} - c_2)(\sqrt{t} - c_3)(-\sqrt{t} - c_3)(\sqrt{t} - c_4)(-\sqrt{t} - c_4) \\ &= (-1)^4 4(t - c_1^2)(t - c_2^2)(t - c_3^2)(t - c_4^2) \end{aligned}$$

so the polynomial with roots c_1^2, c_2^2, c_3^2, c_4^2 may be found by multiplying out the product $f(\sqrt{t})f(-\sqrt{t})$. A short computation produces $f(\sqrt{t})f(-\sqrt{t}) = 4t^4 - 89t^3 + 586t^2 - 1181t + 529$, and this is precisely $g(t)$. Thus, the roots of g are the squares of the roots of f, and we are done.

Solution 2: A somewhat involved calculation will show that $q(t^2+4t+1) = p(t) \cdot (2t^4+33t^3+200t^2+527t+509)$. Upon setting $t = a_i$, we see that $q(a_i^2 + 4a_i + 1) = 0$, meaning that the values $a_i^2 + 4a_i + 1$ are among the roots of $q(t)$. To show that $b_i = a_i^2 + 4a_i + 1$, we need to obtain bounds on the sizes of the values of the a_i. We can compute that $p(-2) > 0$, $p(-1) < 0$, $p(0) > 0$, $p(1) < 0$, and $p(2) > 0$, so by the intermediate value theorem, p has roots in each of the intervals $(-2, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, 2)$. Since $a^2+4a+1 = (a+2)^2-3$, and the vertex of this parabola is at $a = -2$, we conclude that the values $a_i^2 + 4a_i + 1$ for $1 \le i \le 4$ are arranged in increasing order, meaning that $a_i^2 + 4a_i + 1 = b_i$ as claimed.

Solution 3: We expand out the polynomial $\Pi(t) = 4[t-(a_1^2+4a_1+1)][t-(a_2^2+4a_2+1)][t-(a_3^2+4a_3+1)][t-(a_4^2+4a_4+1)]$ and show it is equal to $q(t)$. Expanding shows that $\Pi(t) = 4t^4-4\Sigma_1 t^3+4\Sigma_2 t^2-4\Sigma_3 t+4\Sigma_4$, where Σ_1, Σ_2, Σ_3, Σ_4 are the four symmetric polynomials[1] of $a_1^2+4a_1+1$, $a_2^2+4a_2+1$, $a_3^2+4a_3+1$, $a_4^2+4a_4+1$. A lengthy algebraic computation will allow us to write these polynomials in terms of the symmetric polynomials σ_1, σ_2, σ_3, and σ_4 of a_1, a_2, a_3, and a_4, where $p(t)/2 = (t - a_1)(t - a_2)(t - a_3)(t - a_4) = t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4$. The result is

$$\begin{aligned} \Sigma_1 &= 4 + 4\sigma_1 + \sigma_1^2 - 2\sigma_2 \\ \Sigma_2 &= 6 + 12\sigma_1 + 3\sigma_1^2 + 10\sigma_2 + 4\sigma_1\sigma_2 + \sigma_2^2 - 12\sigma_3 - 2\sigma_1\sigma_3 + 2\sigma_4 \\ \Sigma_3 &= 4 + 12\sigma_1 + 3\sigma_1^2 + 26\sigma_2 + 8\sigma_1\sigma_2 + 2\sigma_2^2 + 40\sigma_3 + 12\sigma_1\sigma_3 + 4\sigma_2\sigma_3 \\ &\quad +\sigma_3^2 - 60\sigma_4 - 12\sigma_1\sigma_4 - 2\sigma_2\sigma_4 \\ \Sigma_4 &= 1 + 4\sigma_1 + \sigma_1^2 + 14\sigma_2 + 4\sigma_1\sigma_2 + \sigma_2^2 + 52\sigma_3 + 14\sigma_1\sigma_3 + 4\sigma_2\sigma_3 + \sigma_3^2 \\ &\quad +194\sigma_4 + 52\sigma_1\sigma_4 + 14\sigma_2\sigma_4 + 4\sigma_3\sigma_4 + \sigma_4^2 \end{aligned}$$

and upon setting $\sigma_1 = \dfrac{1}{2}$, $\sigma_2 = -2$, $\sigma_3 = -\dfrac{1}{2}$, $\sigma_4 = \dfrac{1}{2}$, we obtain $\Sigma_1 = \dfrac{41}{4}$, $\Sigma_2 = \dfrac{1}{4}$, $\Sigma_3 = -91$, $\Sigma_4 = \dfrac{181}{4}$. Thus, $\Pi(t) = 4t^4 - 4\Sigma_1 t^3 + 4\Sigma_2 t^2 - 4\Sigma_3 t + 4\Sigma_4 = 4t^4 - 41t^3 + t^2 + 364t + 181 = q(t)$, so the values $a_i^2 + 4a_i + 1$ are the roots of $q(t)$. The verification that $b_i = a_i^2 + 4a_i + 1$ then follows as in Solution 2.

[1] The kth symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$ is the sum of the $\binom{n}{k}$ possible products of k of the terms x_i. Thus, for example, $\sigma_3(x_1, x_2, x_3, x_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$.