Vermont Mathematics Talent Search, Solutions to Test 3, 2017-2018

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1. Nick has 2 identical cups: the first cup is full of water, while the second is empty. He pours half the water from the first into the second. Then, on transfer 2, he pours $1/3$ of the water in the 2nd cup back into the first cup. He repeats this, alternating cups, pouring $1/(i+1)$ of the water in a cup back into the other cup on the ith transfer. What fraction of the water is in the first cup just after the 2018th transfer?

   Answer: 1010/2019.

   Solution: Suppose the first cup contains 1 unit of water.

   After transfer 1, cup 2 contains $\frac{1}{2}$ unit of water, and cup 1 also contains $\frac{1}{2}$ unit of water.
   After transfer 2, cup 2 contains $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ unit of water, so that cup 1 contains $\frac{2}{3}$ unit of water.
   After transfer 3, cup 1 has $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$ unit of water, so that cup 2 also contains $\frac{1}{2}$ unit of water.
   After transfer 4, cup 2 contains $\frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$ unit of water, so that cup 1 contains $\frac{3}{5}$ unit of water.
   After transfer 5, cup 1 has $\frac{3}{5} \cdot \frac{5}{6} = \frac{1}{2}$ unit of water, so that cup 2 also contains $\frac{1}{2}$ unit of water.
   In a similar way, after every odd-numbered transfer, we see that both cups will contain half the water. Thus, after transfer 2017, both cups again have $\frac{1}{2}$ unit of water, and after transfer 2018, cup 2 will have $\frac{1}{2} \cdot \frac{2018}{2019} = \frac{1009}{2019}$ of the water while cup 1 will have $1 - \frac{1009}{2019} = \boxed{\frac{1010}{2019}}$ of the water.

2. Suppose you are a sportsbook taking 100 bets for who will win the Super Bowl. If each bettor picks exactly one team from the Vikings, Jaguars, Eagles and Patriots, how many different combinations of wagers are possible for the 100 bets you are taking? Here are some examples of possible combinations. You do not distinguish between individual bettors.
   (i) V = 30, J = 15, E = 40, P = 15.
   (ii) V = 8, J = 32, E = 60, P = 0.
   (iii) V = 0, J = 0, E = 0, P = 100.

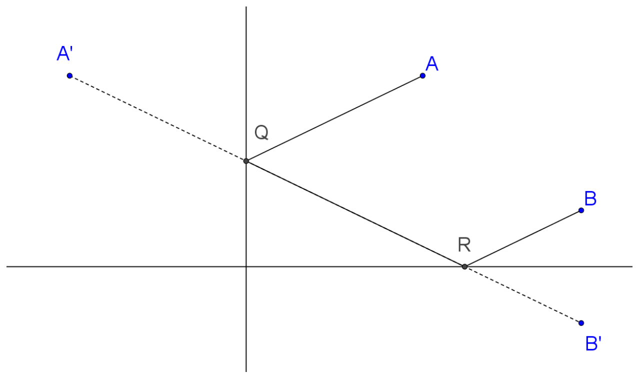
   Answer: 176851.

   Solution: We are seeking to count the number of distinct quadruples (V, J, E, P) with $V + J + E + P = 100$. Consider the problem of arranging 100 identical stars and 3 identical separators in a row: there are $\binom{103}{3}$ such arrangements, since there are 103 possible locations for the 3 separators. If we let V be the number of stars before the first separator, J be the number of stars between the first and second separators, E be the number of stars between the second and third separators, and P be the number of stars after the last separator, then each arrangement of stars and separators yields exactly one quadruple (V, J, E, P) with $V + J + E + P = 100$. Conversely, each such quadruple corresponds to a distinct arrangement of stars and separators, so the number of quadruples is $= \frac{103 \cdot 102 \cdot 101}{3 \cdot 2} = \boxed{176851}$.

3. What is the length of the shortest path AQRB in the plane, where A = (14, 12), B = (29, 1), Q lies on the y-axis, and R lies on the x-axis?

Answer: $\sqrt{2018}$.

Solution: Let $A' = (-14, 12)$ and $B' = (29, -1)$ be the reflections of A and B across the y- and x-axes respectively:



Observe that the length of segment $A'Q$ is the same as the length of segment AQ, and, similarly, the length of segment $B'P$ is the same as the length of segment BP. Therefore, the length of path $AQRB$ is the same as the length of path $A'QRB'$. Since the line segment $A'B'$ crosses both the y- and x-axes, we see that the minimal possible length of this path occurs when $A'QRB'$ is the straight line from $(-14, 12)$ to $(29, -1)$. The length of this minimal path is then $\sqrt{(29-(-14))^2 + ((-1)-12)^2} = \sqrt{43^2 + 13^2} = \boxed{\sqrt{2018}}$.

Remark 1: The path of minimal length occurs when $A'B'$ is the line $y + 1 = -\frac{13}{43}(x - 29)$, which intersects the axes at $Q = (0, \frac{334}{43})$ and $R = (\frac{334}{13}, 0)$.

Remark 2: A direct approach to this problem would be to let $Q = (0, y)$ and $R = (x, 0)$, and then try to minimize the path length $d(x, y) = \sqrt{14^2 + (12-y)^2} + \sqrt{x^2 + y^2} + \sqrt{(29-x)^2 + 1^2}$. However, even by using calculus or other methods, this approach is extremely messy and complicated.

4. If $0 < \theta < \frac{\pi}{2}$ and $\frac{1 + 2\sin(\theta) + 3\sin^2(\theta) + 4\sin^3(\theta) + \cdots}{1 + 2\cos(\theta) + 3\cos^2(\theta) + 4\cos^3(\theta) + \cdots} = \frac{4}{81}$, find the value of $1 + 2\tan(\theta) + 3\tan^2(\theta) + 4\tan^3(\theta) + \cdots$.

Answer: 225/49.

Solution: If $|x| < 1$, we can write

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \cdots &= (1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots) + (x^2 + x^3 + \cdots) + \cdots \\ &= \frac{1}{1-x} + \frac{x}{1-x} + \frac{x^2}{1-x} + \cdots \\ &= \frac{1}{1-x}(1 + x + x^2 + \cdots) = \frac{1}{(1-x)^2}. \end{aligned}$$

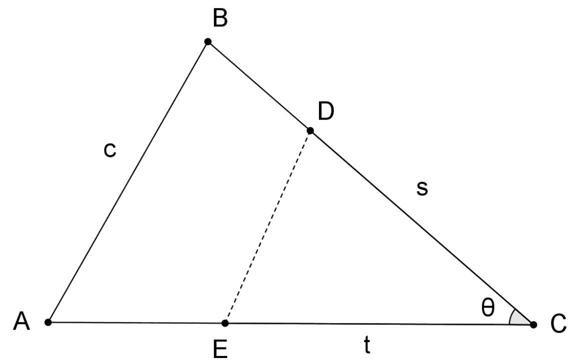
Thus, by applying this formula for $x = \sin(\theta)$ and $x = \cos(\theta)$, we see that the given relation is equivalent to $\frac{1/(1-\sin\theta)^2}{1/(1-\cos\theta)^2} = \frac{4}{81}$. Since $1 - \sin\theta$ and $1 - \cos\theta$ are both nonnegative, we may take the square root of both sides to obtain $\frac{1-\cos\theta}{1-\sin\theta} = \frac{2}{9}$, so that $2\sin\theta = 9\cos\theta - 7$. Squaring both sides yields $4(1-\cos^2\theta) = 81\cos^2\theta - 126\cos\theta + 49$, so that $85\cos^2\theta - 126\cos\theta + 45 = 0$. This factors as $(17\cos\theta - 15)(5\cos\theta - 3)$, so $\cos\theta = 3/5$ or $\cos\theta = 15/17$. However, if $0 < \theta < \pi/2$, then $\cos\theta = 3/5$ implies $\sin\theta = 4/5$, but this does not satisfy the equation $\frac{1/(1-\sin\theta)^2}{1/(1-\cos\theta)^2} = \frac{4}{81}$. Thus, $\cos\theta = 15/17$ and then $\sin\theta = 8/17$, so that $\tan\theta = 8/15$. The desired expression is then $1 + 2\tan\theta + 3\tan^2\theta + \cdots = \frac{1}{(1-\tan\theta)^2} = \frac{1}{(7/15)^2} = \boxed{\frac{225}{49}}$.

Remark: The original relation makes sense for $-\pi/2 < \theta < \pi/2$, and inside this interval there is a second solution with $(\cos\theta, \sin\theta) = (3/5, -4/5)$. However, this value of θ has $\tan\theta = -4/3$, so the infinite series $1 + 2\tan(\theta) + 3\tan^2(\theta) + 4\tan^3(\theta) + \cdots$ does not converge.

5. Triangle ABC has AB = 22, AC = 24, and BC = 26, and line l bisects both the perimeter and area of ABC. If line l intersects triangle ABC in points D and E, find all possible lengths of segment DE.

Answer: $DE = 12\sqrt{2}$ or $12\sqrt{3}$. [Note: 24 is not a possible length.]

Solution: Label the sides of ABC as a, b, and c, and suppose that segment DE intersects the sides of lengths a and b, cutting off pieces of lengths s and t as shown in the diagram below.



The area of triangle ABC is $\frac{1}{2}ab\sin(\theta)$, while the area of triangle CDE is $\frac{1}{2}st\sin(\theta)$: therefore, since DE bisects the area of ABC, we have $st = \frac{1}{2}ab$. Likewise, since DE bisects the perimeter of ABC, we see that $s+t = \frac{1}{2}(a+b+c)$. Therefore, we see $s^2 + \frac{1}{2}ab = s^2 + st = \frac{1}{2}(a+b+c)s$, so s satisfies the quadratic equation $s^2 - \frac{1}{2}(a+b+c)s + \frac{1}{2}ab = 0$. Solving yields $s = \frac{1}{4}(a+b+c) \pm \frac{1}{4}\sqrt{(a+b+c)^2 - 8ab}$, and then $t = \frac{1}{4}(a+b+c) \mp \frac{1}{4}\sqrt{(a+b+c)^2 - 8ab}$.

For the three possible cases $c = 22$, $c = 24$, and $c = 26$, we obtain $(s,t) = 18 \pm 2\sqrt{3}$, $18 \pm \sqrt{38}$, and $18 \pm 2\sqrt{15}$ respectively. Since $18 + 2\sqrt{3} < 22$, we obtain two bisector segments with $c = 22$ (cutting off a piece of length $18 + 2\sqrt{3}$ from either AC or BC and a piece of length $18 - 2\sqrt{3}$ from the other).

Since $24 < 18 + \sqrt{38} < 25$, we obtain a single bisector segment with $c = 24$ (cutting off a piece of length $18 + \sqrt{38}$ from BC and a piece of length $18 - \sqrt{38}$ from AB).

Finally, since $25 < 18 + 2\sqrt{15} < 26$, it is not possible to have a bisector segment with $c = 26$, since it is not possible to cut off a segment of length $18 + 2\sqrt{15}$ from either AB or AC, as they are both too short.

By the Law of Cosines in triangles ABC and DEF, we see that $DE^2 = s^2 + t^2 - 2st\cos(\theta)$, and $c^2 = a^2 + b^2 - 2ab\cos(\theta)$: therefore, $DE^2 = (s+t)^2 - 2st - 2st\cos\theta = \frac{1}{4}(a+b+c)^2 - ab - ab \cdot \frac{a^2+b^2-c^2}{2ab} = \frac{1}{4}(a+b+c)^2 - ab - \frac{a^2+b^2-c^2}{2}$.

For the two possible values $c = 22$ and $c = 24$, we obtain $DE^2 = 288$ and 432 respectively, yielding the two possible lengths $DE = \boxed{12\sqrt{2},\ 12\sqrt{3}}$.

Remark: The original solution to this problem contained a calculation error that resulted in not eliminating the invalid case $c = 26$.

6. Let $p(x)$ be a monic polynomial with integer coefficients. We call a triangle p-special if its 3 vertices all have integer coordinates and lie on the graph of $y = p(x)$, and we say the positive integer n is p-special if there is a p-special triangle whose area is n.

   (a) Show that there exists a polynomial p of degree 3 such that the p-special integers are precisely the positive multiples of 3.

   (b) Determine, with proof, whether there exists a polynomial p such that every positive integer is p-special.

Motivation: Suppose that the three points are $(a, p(a))$, $(b, p(b))$, and $(c, p(c))$, where a, b, c are integers. Then by the "shoelace formula", the area of the triangle is the absolute value of the quantity

$$B_p(a,b,c) = \frac{1}{2}\left(a\,[p(b)-p(c)] + b\,[p(c)-p(a)] + c\,[p(a)-p(b)]\right).$$

The set of p-special integers are then those of the form $|B_p(a,b,c)|$ for some integers a, b, c. Notice that if $a = b$, $b = c$, or $a = c$, then the area is zero, so the function $B_p(a,b,c)$ will always factor to include the terms $a - b$, $a - c$, and $b - c$. (In particular, since one of these three values must be even, the area of a p-special triangle is always an integer.)

Solution (a): We claim that the polynomial $p(x) = x^3$ has the desired property. To see this, observe that by the remark above, the area of the triangle is

$$\frac{1}{2}\left|a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)\right| = \frac{1}{2}\left|(a-b)(a-c)(b-c)(a+b+c)\right|.$$

Notice that for any integers a, b, and c, this quantity is always divisible by 3: if any of a, b, c are equal modulo 3, then $a - b$, $a - c$, or $b - c$ will be divisible by 3, while if a, b, c are all different modulo 3, then $a + b + c \equiv 0 + 1 + 2 \equiv 0 \pmod 3$ is divisible by 3. Therefore, any p-special integer must be divisible by 3. But now if we take $a = b - 1$ and $c = b + 1$ for any positive integer b, the area reduces to $\frac{1}{2}|1 \cdot 2 \cdot 1 \cdot 3b| = 3b$. Therefore, every positive integer multiple of 3 is p-special for $p(x) = x^3$, so we conclude that the p-special integers are precisely the positive multiples of 3.

Solution (b): The result is true: for example, the polynomial $p(x) = x^3 + x$ has the property that every positive integer is p-special. To show this, again by the remark preceding part (a), we wish to analyze the function

$$A(a,b,c) = \frac{1}{2}\left|a(b^3 + b - c^3 - c) + b(c^3 + c - a^3 - a) + c(a^3 + a - b^3 - b)\right| = \frac{1}{2}\left|(a-b)(a-c)(b-c)(a+b+c+1)\right|.$$

We show that every positive integer n can be written as $A(a,b,c)$ for suitable a, b, and c:

If $n \equiv 1 \pmod 3$, so that $n = 3k + 1$ for an integer $k \geq 0$, then we have $A(k+1,\ k,\ k-1) = \frac{1}{2}|1 \cdot 2 \cdot 1 \cdot (3k+1)| = 3k + 1$.

If $n \equiv 2 \pmod 3$, so that $n = 3k + 2$ for an integer $k \geq 0$, then we have $A(-k,\ -k-1,\ -k-2) = \frac{1}{2}|1 \cdot 2 \cdot 1 \cdot (-3k-2)| = 3k + 2$.

If $n \equiv 3 \pmod 9$, so that $n = 9k + 3$ for an integer $k \geq 0$, then we have $A(-k+1,\ -k-1,\ -k-2) = \frac{1}{2}|2 \cdot 3 \cdot 1 \cdot (-3k-1)| = 9k + 3$.

If $n \equiv 6 \pmod 9$, so that $n = 9k + 6$ for an integer $k \geq 0$, then we have $A(k+2,\ k,\ k-1) = \frac{1}{2}|2 \cdot 3 \cdot 1 \cdot (3k+2)| = 9k + 6$.

If $n \equiv 0 \pmod 9$, so that $n = 9k$ for an integer $k \geq 0$, then we have $A(k+1,\ k,\ k-2) = \frac{1}{2}|1 \cdot 3 \cdot 2 \cdot (3k)| = 9k$.

Each positive integer falls into one of these five classes, so we conclude that every positive integer is p-special for $p(x) = x^3 + x$.

Remark 1: Although many solvers correctly wrote down a formula for the area (including the absolute value), many proofs were submitted claiming that there was no polynomial that would work for part (b). In fact, the presence of the absolute value signs in the area formula is crucial, because the values of $B_p(a,b,c)$ can take values in only one of the nonzero residue classes modulo 3 (as proven by several solvers).

Remark 2: With notation as above, observe that for any polynomials p_1 and p_2, we have $B_{p_1+p_2}(a,b,c) = B_{p_1}(a,b,c) + B_{p_2}(a,b,c)$, and because the three points are collinear if p is a linear function, we see that $B_p(a,b,c)$ is zero for $p(x) = 1$ and for $p(x) = x$. Thus, $B_p(a,b,c)$ depends only on the terms in $p(x)$ of degree 2 and higher.

Remark 3: By the previous remark, we can see that $B_p(a,b,c)$ for $p(x) = x^2 + rx + s$ will be independent of r and s, and a short calculation will produce $B_p(a,b,c) = \frac{1}{2}(a-b)(b-c)(c-a)$. It can be shown by a short case analysis that neither 4 nor 12 can be written as $\left|\frac{1}{2}(a-b)(b-c)(c-a)\right|$ for any integers a, b, and c, so no quadratic polynomial will work in part (a) or part (b). By a similar analysis to that given above, it can be shown that for $p(x) = x^3 + rx^2 + sx + t$, then the p-special integers are the multiples of 3 (if r is divisible by 3), or all integers (if r is not divisible by 3). The authors do not know whether there are any polynomials of degree larger than 3 whose p-special integers can be easily characterized.