Vermont Mathematics Talent Search, Solutions to Test 4, 2017-2018

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. Suppose that $p(x)$ is a polynomial all of whose coefficients are either 0 or 1. If $p(\sqrt{2}) = 20 + 18\sqrt{2}$, find the value of $p(2)$.

   Answer: 792.

   Solution: Suppose $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$: then $p(\sqrt{2}) = (a_0 + 2a_2 + 4a_4 + 8a_6 + \cdots) + (a_1 + 2a_3 + 4a_5 + 8a_7 + \cdots)\sqrt{2}$. Since all of the coefficients are either 0 or 1 and $\sqrt{2}$ is irrational, we must have $a_0 + 2a_2 + 4a_4 + 8a_6 + \cdots = 20$ and $a_1 + 2a_3 + 4a_5 + 8a_7 + \cdots = 18$. But now observe that because each coefficient is either 0 or 1, the left-hand side of each equation is a base-2 expansion, and so we can read off the appropriate coefficients by writing 20 and 18 in base 2. We see that $20 = 10100_2$ and $18 = 10010_2$, so $a_3 = a_4 = a_8 = a_9 = 1$ and all other coefficients are zero. Thus, $p(x) = x^3 + x^4 + x^8 + x^9$, and so $p(2) = 2^3 + 2^4 + 2^8 + 2^9 = \boxed{792}$.

2. The graph of the equation $47x + 43y - 3 = 2018$ is drawn on graph paper with each square representing one unit in each direction. The grid lines begin at the origin, and are vertical and horizontal. How many of the 1×1 squares have interiors lying entirely below the graph and entirely in the first quadrant?

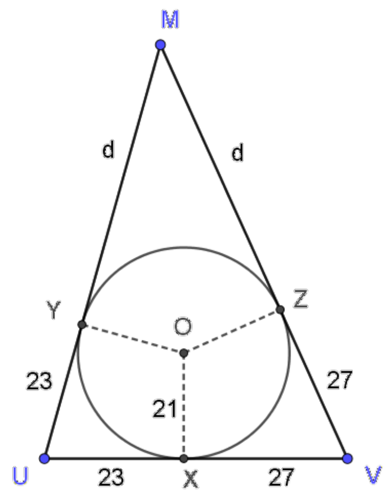
   Answer: 966.

   Solution: The equation describes the line $\frac{x}{43} + \frac{y}{47} = 1$, so it intersects the axes at $(0, 47)$ and $(43, 0)$. The line's path from $(0, 47)$ to $(43, 0)$ then crosses 47 horizontal lines and 43 vertical lines, and since 47 and 43 are relatively prime, none of these crossings occur simultaneously. Therefore, the line passes through the interiors of precisely $47 + 43 + 1 = 91$ of the 1×1 squares. Furthermore, by symmetry, there are exactly as many squares lying entirely below the line in the first quadrant as there are lying entirely above the line below $y = 47$ and to the left of $x = 43$. Therefore, the number of squares lying entirely below the line in the first quadrant is $\frac{1}{2}(47 \cdot 43 - 91) = 21 \cdot 46 = \boxed{966}$.

3. The inscribed circle of triangle UVM is tangent to UV at X, and its radius is 21. Given that UX = 23 and XV = 27, find the perimeter of the triangle.

   Answer: 345.

   Solution 1: Suppose that the center of the circle is at point O and that the circle is tangent to UM at Y and to MV at Z.



Then $UY = UX = 23$, $VX = VZ = 27$, and $MY = MZ = d$ for some d. Then by adding the areas of triangles MOU, MOV, and MUV, we see that $[UVM] = [MOU] + [MOV] + [MUV] = 21(50 + d)$.
Since the semiperimeter of triangle UVM is $50+d$, by Heron's formula we also have $[UVM] = \sqrt{(50 + d)(27)(23)(d)}$.
By setting these two expressions equal and squaring both sides, we obtain $21^2(50 + d)^2 = (50 + d)(27)(23)(d)$, so that $\frac{50+d}{d} = \frac{27 \cdot 23}{21^2}$, whence $d = \frac{245}{2}$. The perimeter of triangle UVM is then $100 + 2d = \boxed{345}$.

Solution 2: We use the same notation as in Solution 1. Observe that $\pi - \angle MOY = \angle UOX + \angle VOZ$, and furthermore, we have $\tan(\angle MOY) = \frac{d}{21}$, $\tan(\angle UOX) = \frac{23}{21}$, and $\tan(\angle VOZ) = \frac{27}{21}$.
Thus, taking the tangent of both sides of $\pi - \angle MOY = \angle UOX + \angle VOZ$ and applying the tangent addition formula yields $-\frac{d}{21} = \frac{23/21 + 27/21}{1 - (23/21)(27/21)} = \frac{50/21}{-180/21^2}$.
Thus, we obtain $d = \frac{5}{18} \cdot 21^2 = \frac{245}{2}$, and then the perimeter of triangle UVM is $100 + 2d = \boxed{345}$.

4. We say that a sequence of distinct positive integers $a_1, a_2, \dots, a_n$ forms a *VMTS 25th anniversary sequence* if $\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right)\cdots\left(1 - \frac{1}{a_n}\right) = \frac{25}{2018}$.

   (a) Show that there exists a VMTS 25th anniversary sequence with 81 terms.

   (b) Show that there does not exist a VMTS 25th anniversary sequence with 80 terms.

Solution (a): Note that we can write the telescoping product

$$\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a+1}\right)\cdots\left(1 - \frac{1}{b}\right) = \frac{a-1}{a} \cdot \frac{a}{a+1} \cdot \cdots \cdot \frac{b-1}{b} = \frac{a-1}{b}.$$

Then we can easily verify that

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{80}\right)\left(1 - \frac{1}{126}\right)\left(1 - \frac{1}{1009}\right) = \frac{1}{80} \cdot \frac{125}{126} \cdot \frac{1008}{1009} = \frac{25}{2018}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{72}\right)\left(1 - \frac{1}{76}\right)\cdots\left(1 - \frac{1}{84}\right)\left(1 - \frac{1}{1009}\right) = \frac{1}{72} \cdot \frac{75}{84} \cdot \frac{1008}{1009} = \frac{25}{2018}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{74}\right)\left(1 - \frac{1}{76}\right)\cdots\left(1 - \frac{1}{81}\right)\left(1 - \frac{1}{112}\right)\left(1 - \frac{1}{1009}\right) = \frac{1}{74} \cdot \frac{75}{81} \cdot \frac{111}{112} \cdot \frac{1008}{1009} = \frac{25}{2018}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{80}\right)\left(1 - \frac{1}{113}\right)\left(1 - \frac{1}{14126}\right) = \frac{1}{80} \cdot \frac{112}{113} \cdot \frac{14125}{14126} = \frac{25}{2018}$$

Thus, the sequences $\{2, 3, \dots, 80, 126, 1009\}$, $\{2, 3, \dots, 72, 76, \dots, 84, 1009\}$, $\{2, 3, \dots, 74, 76, \dots, 81, 112, 1009\}$ and $\{2, 3, \dots, 80, 113, 14126\}$ are each VMTS 25th anniversary sequences with 81 terms.

Solution (b): Suppose that we had 80 distinct positive integers $a_1, a_2, \dots, a_n$ such that

$$\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\cdots\left(1-\frac{1}{a_{80}}\right)=\frac{25}{2018}.$$

Since $2018 = 2 \cdot 1009$ and 1009 is prime, at least one of the integers a_i must be divisible by 1009. Then the smallest possible positive value for the product $\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\cdots\left(1-\frac{1}{a_{80}}\right)$ is

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{80}\right)\left(1-\frac{1}{1009}\right)=\frac{1}{80}\cdot\frac{1008}{1009}>\frac{1000}{80\cdot 1009}=\frac{25}{2018}.$$

But this is a contradiction, so there exists no such sequence.

5. Three data scientists eat lunch in the cafeteria every day. They arrive there independently, at random times uniformly chosen between 11:00 AM and 1:00 PM, and stay for exactly 20 minutes. What is the probability that all three data scientists are in the cafeteria simultaneously at some point?

Answer: 2/27.

Solution: Suppose the arrival times are $x/2$, $y/2$, and $z/2$ hours after 11:00 AM, so that x, y, and z are chosen uniformly from the interval $[0,1]$. Then the three data scientists will all be in the cafeteria simultaneously if and only if $|x-y|$, $|x-z|$, and $|y-z|$ are all less than or equal to $1/6$. The resulting inequalities $0 \le x, y, z \le 1$ and $|x-y|, |x-z|, |y-z| \le 1/6$ describe a solid R in 3-dimensional space whose volume is the desired probability.

By symmetry, R decomposes into six pieces each with equal volume corresponding to the 6 possible orderings of x, y, and z. Thus, it suffices to compute the volume of the portion of the region R' with $x \le y \le z$, which is described by the simpler inequalities $0 \le x \le 1$ and $x \le y \le z \le x + 1/6$. We can see (per the picture above) R' is composed of two pieces: the piece for $0 \le x \le 5/6$ is a slanted triangular prism, while the piece for $5/6 \le x \le 1$ is a triangular pyramid.

The slanted triangular prism has height $5/6$ and a right-triangular base having two legs of length $1/6$ (hence area $1/72$): thus, its volume is $\frac{5}{6}\cdot\frac{1}{72}=\frac{5}{432}$. The triangular pyramid has height $1/6$ and a right-triangular base having two legs of length $1/6$ (hence area $1/72$): thus, its volume is $\frac{1}{3}\cdot\frac{1}{6}\cdot\frac{1}{72}=\frac{1}{1296}$. Thus, the total volume of R' is $\frac{5}{432}+\frac{1}{1296}=\frac{1}{81}$ hence the volume of R is $6\cdot\frac{1}{81}=\boxed{\frac{2}{27}}$.

Remark: In a similar way (though requiring a more careful analysis of the d-dimensional region), it can be shown that for d data scientists the probability that all of them are in the cafeteria simultaneously is $d(1/6)^{d-1}+(d-1)(1/6)^d$.

6. Given that there exist unique rational numbers a, b, and c for which $\cos^7(\pi/7) = a\cos(\pi/7) + b\cos(3\pi/7) + c\cos(5\pi/7)$, find the ordered triple (a, b, c).

Answer: $(a,b,c)=\left(\frac{33}{64},\frac{19}{64},\frac{5}{64}\right)$.

Solution 1: Notice that $\cos(\pi/7)=\frac{1}{2}(\zeta_7+\zeta_7^{-1})$ where $\zeta_7 = e^{2\pi i/7} = \cos(2\pi/7) + i\sin(2\pi/7)$ is a primitive 7th root of unity, which in particular has the property that $\zeta_7^7 = 1$. Since $\zeta_7 \ne 1$, by factoring we also see $1+\zeta_7+\zeta_7^2+\zeta_7^3+\zeta_7^4+\zeta_7^5+\zeta_7^6=0$, which (upon rearranging and using the fact that $\zeta_7^{-k}=\zeta_7^{7-k}$ for $k = 1, 3, 5$) implies that $1+\zeta_7^5+\zeta_7^3+\zeta_7+\zeta_7^{-1}+\zeta_7^{-3}+\zeta_7^{-5}=0$.

Furthermore, observe that $\cos(3\pi/7)=\frac{1}{2}(\zeta_7^3+\zeta_7^{-3})$ and $\cos(5\pi/7)=\frac{1}{2}(\zeta_7^5+\zeta_7^{-5})$. By the binomial theorem, we can then compute

$$\begin{aligned}
\cos^7(\pi/7) &= \frac{1}{2^7}(\zeta_7^7 + 7\zeta_7^5 + 21\zeta_7^3 + 35\zeta_7 + 35\zeta_7^{-1} + 21\zeta_7^{-3} + 7\zeta_7^{-5} + \zeta_7^{-7}) \\
&= \frac{1}{2^7}(1 + 7\zeta_7^5 + 21\zeta_7^3 + 35\zeta_7 + 35\zeta_7^{-1} + 21\zeta_7^{-3} + 7\zeta_7^{-5} + 1) \\
&= \frac{1}{2^7}(5\zeta_7^5 + 19\zeta_7^3 + 33\zeta_7 + 33\zeta_7^{-1} + 19\zeta_7^{-3} + 5\zeta_7^{-5}) \\
&= \frac{33}{64}(\zeta_7 + \zeta_7^{-1}) + \frac{19}{64}(\zeta_7^3 + \zeta_7^{-3}) + \frac{5}{64}(\zeta_7^5 + \zeta_7^{-5}) \\
&= \frac{33}{64}\cos(\pi/7) + \frac{19}{64}\cos(3\pi/7) + \frac{5}{64}\cos(5\pi/7)
\end{aligned}$$

and therefore we can take $(a, b, c) = \boxed{\left(\frac{33}{64}, \frac{19}{64}, \frac{5}{64}\right)}$.

Solution 2: We suppose that $\cos^n(\pi/7) = a_n \cos(\pi/7) + b_n \cos(3\pi/7) + c_n \cos(5\pi/7)$ for rational numbers a_n, b_n, c_n and derive a recurrence relation for the coefficients a_n, b_n, c_n: setting $n = 1$ indicates that we want to take $a_1 = 1$ and $b_1 = c_1 = 0$.

Now, if $\cos^n(\pi/7) = a_n \cos(\pi/7) + b_n \cos(3\pi/7) + c_n \cos(5\pi/7)$ then by using the double-angle and product-to-sum identities, along with the identity $\cos(\pi/7) + \cos(3\pi/7) + \cos(5\pi/7) = \frac{1}{2}$ which can be derived from them, we have

$$\begin{aligned}
\cos^{n+1}(\pi/7) &= a_n \cos^2(\pi/7) + b_n \cos(3\pi/7)\cos(\pi/7) + c_n \cos(5\pi/7)\cos(\pi/7) \\
&= a_n \frac{\cos(2\pi/7) + 1}{2} + b_n \frac{\cos(4\pi/7) + \cos(2\pi/7)}{2} + c_n \frac{\cos(6\pi/7) + \cos(4\pi/7)}{2} \\
&= a_n \frac{2\cos(\pi/7) + 2\cos(3\pi/7) + \cos(5\pi/7)}{2} + b_n \frac{-\cos(3\pi/7) - \cos(5\pi/7)}{2} \\
&\quad + c_n \frac{-\cos(\pi/7) - \cos(3\pi/7)}{2} \\
&= \left[a_n - \frac{1}{2}c_n\right]\cos(\pi/7) + \left[a_n - \frac{1}{2}b_n - \frac{1}{2}c_n\right]\cos(3\pi/7) + \left[\frac{1}{2}a_n - \frac{1}{2}b_n\right]\cos(5\pi/7)
\end{aligned}$$

from which we deduce $a_{n+1} = a_n - \frac{1}{2}c_n$, $b_{n+1} = a_n - \frac{1}{2}b_n - \frac{1}{2}c_n$, and $c_{n+1} = \frac{1}{2}a_n - \frac{1}{2}b_n$. We can then calculate a_7, b_7, c_7 recursively:

n	1	2	3	4	5	6	7
a_n	1	1	3/4	3/4	5/8	19/32	33/64
b_n	0	1	1/4	5/8	5/16	7/16	19/64
c_n	0	1/2	0	1/4	1/16	5/32	5/64

and therefore we conclude that $(a, b, c) = \boxed{\left(\frac{33}{64}, \frac{19}{64}, \frac{5}{64}\right)}$.

Remark: The problem statement implies that the coefficients (a, b, c) are unique. Here is a brief explanation why this is true: let $x = \cos(\pi/7)$ so that $\cos(5\pi/7) = -\cos(2\pi/7) = 2x^2 - 1$ and $\cos(3\pi/7) = 4x^3 - 3x$. As noted in Solution 2, we have the relation $\cos(\pi/7) + \cos(3\pi/7) + \cos(5\pi/7) = \frac{1}{2}$, so that $8x^3 - 4x^2 - 4x + 1 = 0$. If there were another choice of coefficients (a, b, c), then subtracting and substituting $\cos(5\pi/7) = -\cos(2\pi/7)$ and $\cos(3\pi/7) = 1 - \cos(\pi/7) + \cos(2\pi/7)$ would yield a relation of the form $a' + b'\cos(\pi/7) + c'\cos(2\pi/7) = 0$ for nonzero rational numbers a', b', c': thus, x would be the root of a nonzero quadratic polynomial with rational coefficients. However, this is impossible, because the polynomial $p(t) = 8t^3 - 4t^2 - 4t + 1$ is irreducible over the rational numbers (i.e., it does not factor as the product of two polynomials with rational coefficients of smaller positive degree) since it has no rational roots. In fact, it essentially follows from the discussion above that the roots of the polynomial $8t^3 - 4t^2 - 4t + 1 = 0$ are precisely $\cos(\pi/7)$, $\cos(3\pi/7)$, $\cos(5\pi/7)$.