1. A large rectangle is divided into three smaller rectangles and a square as shown in the diagram. If the three nonsquare rectangles have areas 96, 120, and 180 (in some order), find all possible side lengths for the square.

Answer: 8, 12, 15.

Solution: Suppose the square has side length \(s\). If the two segments on the interior are divided into pieces of lengths \(a, s\) and \(b, s\) respectively, then the areas of the four smaller rectangles (including the square) are \(ab\), \(as\), \(bs\), and \(s^2\). The given information then says that \(ab\), \(as\), and \(bs\) are equal to 96, 120, and 180 in some order. Noting that \(s^2 = \frac{as \cdot bs}{180}\), we see that of the following three possibilities must hold:

\[
\begin{align*}
\text{if } s^2 &= \frac{96 \cdot 120}{180} = 64 \text{ so that } s = 8, \\
\text{or } s^2 &= \frac{96 \cdot 180}{120} = 144 \text{ so that } s = 12, \\
\text{or } s^2 &= \frac{120 \cdot 180}{96} = 225 \text{ so that } s = 15.
\end{align*}
\]

It is not hard to verify that each of these three possibilities can actually occur: 
\((a, b, s) = (8, 12, 15)\) yields 
\((ab, as, bs, s^2) = (96, 120, 180, 225)\), while 
\((a, b, s) = (8, 15, 12)\) yields 
\((ab, as, bs, s^2) = (120, 96, 180, 225)\), and 
\((a, b, s) = (12, 15, 8)\) yields 
\((ab, as, bs, s^2) = (180, 96, 120, 64)\). Therefore, the possible side lengths are \(s = 8, 12, 15\).

2. Kat writes down the units digits of the numbers \(1^1, 2^2, 3^3, \ldots, 2016^{2016}, 2017^{2017}, 2018^{2018}\), obtaining a sequence \(1, 4, 7, \ldots, 6, 7, 4\). What is the sum of the numbers in Kat’s sequence?

Answer: 9485.

Solution: Using a computer or otherwise, the first 20 terms of the sequence are \(\{1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0\}\), and the next 20 terms are also \(\{1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0\}\). This suggests that the sequence repeats every 20 terms. In fact, this is true: for any digit \(d\), first observe that \(d^5 \equiv d\pmod{10}\), either by a direct check or by combining the two observations that \(d^5 \equiv d\pmod{2}\) and \(d^5 \equiv d\pmod{5}\). Then we see that \((d+20)^{d+20} \equiv d^{d+20} \equiv d^d\pmod{10}\), by applying the relation \(d^5 \equiv d\pmod{10}\) five times. Hence, the sequence repeats every 20 terms as claimed.

Then the terms 1 through 2000 consist of 100 copies of \(\{1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0\}\), with total sum \(100 \cdot 94\), along with the first 18 terms again, which have sum 85. The total sum of all the terms is therefore \(100 \cdot 94 + 85 = 9485\).

3. Prove that the integers 1 through 32 inclusive can be arranged around a circle so that the sum of each pair of adjacent integers is a perfect square, but that the integers 1 through \(n\) inclusive cannot be so arranged for any integer \(n\) with \(2 \leq n \leq 30\).

Solution: Call two integers “connected” if their sum is a square. In order to arrange the integers in the required way, each integer in \(\{1, 2, \ldots, n\}\) must be connected to at least two other integers, since each integer must have two neighbors on the circle. If \(2 \leq n \leq 6\), then 2 is not connected to any other integer, while if \(7 \leq n \leq 13\), then 2 is connected only to 7. If \(14 \leq n \leq 15\), then 9 is connected only to 7. If
16 \leq n \leq 19$, then $16$ is connected only to $9$. Finally, if $20 \leq n \leq 30$, then $18$ is connected only to $7$. This eliminates any possible $n$ with $2 \leq n \leq 30$.

For $n = 32$, there are several numbers that are only connected to two others (such as $32$, which is only connected to $4$ and $17$). These yield the pieces $4-32-17$, $5-31-18$, $6-30-19$, $7-29-20$, $8-28-21$, $9-27-22$, $8-26-23$, $9-25-24$, $7-18-31$, and $9-16-20$. Several of these pieces must then join together since they share common numbers, and these connections then exclude various other possible adjacencies (e.g., $9-27-22$ must join to $9-16-20$ to form $22-27-9-16-20$, and then this excludes the possibility that $9$ can be adjacent to $7$). Repeating logic of this nature, and avoiding loops, eventually yields a single arrangement (unique up to reversal and reordering):

\[ \begin{array}{cccccc}
26 & 15 & 10 & 11 & 18 & 31 \\
2 & 21 & 19 & 20 & 30 & 17 \\
27 & 16 & 12 & 13 & 14 & 28 \\
22 & 1 & 3 & 4 & 5 & 6 \\
 & 7 & 8 & 9 & 10 & \\
\end{array} \]

**Remark:** When $n = 31$, each integer is connected to at least two others, but it is still not possible to create a satisfactory arrangement. (The only proof of this fact that the problem authors could find involved substantial case analysis similar to the construction for $n = 32$, and for this reason the case $n = 31$ was excluded from the problem.)

4. If $f(n) = \frac{n - 3}{\sqrt{2n - 2} - 2}$, what is the smallest integer $n > 3$ such that $f(f(f(n))))$ is an integer?

**Answer:** $2^{31} + 1 = 2147483649$.

**Solution 1:** Observe that for $n \neq 3$, we have

\[ f(n) = \frac{n - 3}{\sqrt{2n - 2} - 2} \cdot \frac{\sqrt{2n - 2} + 2}{\sqrt{2n - 2} + 2} = \frac{(n - 3) \cdot (\sqrt{2n - 2} + 2)}{2n - 6} = \frac{\sqrt{2n - 2} + 2}{2} = \sqrt{\frac{n - 1}{2}} + 1. \]

Then $f(f(n)) = \sqrt{\frac{n - 1}{8}} + 1$, $f(f(f(n)))) = \sqrt{\frac{n - 1}{128}} + 1$, and $f(f(f(f(n)))) = \sqrt{\frac{n - 1}{512}} + 1$.

Solving $f(f(f(f(n)))) = k$ yields $n = 8^5(k - 1)^{16} + 1$.

If $k = 1$ then we get $n = 1$, which does not have $n > 1$.

If $n = 2$ then we get $n = 8^5 + 1 = 32769$, but this does not work either because $f(f(f(n))) = 3$ and so $f(f(f(f(n))))$ is undefined.

If $k = 3$ then we get $n = 8^5 \cdot 2^{16} + 1 = 2147483649$. This value does work, as $f(n) = 32769$, $f(f(n)) = 129$, $f(f(f(n))) = 9$, and $f(f(f(f(n)))) = 3$ are all defined.

**Solution 2:** Since $f(x)$ is an increasing function when $x > 3$ (as can be shown by a careful algebraic estimate, or using calculus) and $f(x) = 1$ has only the solution $x = 1$, the smallest possible candidate for $n$ would have $f(f(f(n)))) \geq 2$. However, solving $f(f(f(f(n)))) = 2$ yields an invalid solution for $n$ (solving the equation algebraically yields $f(f(n)) = 3$ but this value is not in the domain of $f$), and so we must in fact have $f(f(f(f(n)))) \geq 3$. Solving $f(x) = 3$ yields $x = 9$, so we must have $f(f(f(n))) \geq 9$. Solving $f(x) = 9$ yields $x = 129$, so we must have $f(f(n)) \geq 129$. Solving $f(x) = 129$ yields $x = 32769$, so we must have $f(n) \geq 32769$. Finally, solving $f(n) = 32769$ yields $n = 2147483649$ and the chain of inequalities above shows that no smaller value of $n$ can work.
5. Triangle $ABC$ has $\frac{1}{AB} + \frac{1}{BC} = \cos(B) = \frac{1}{8}$. Find the length of the bisector of angle $B$.

**Answer:** 12.

**Solution:** Let the bisector of angle $B$ intersect $AC$ at $D$, and take $AB = c$, $BC = a$, $BD = d$, and $m\angle ABC = \theta$; then we are given $\frac{1}{c} + \frac{1}{d} = \cos(\theta) = \frac{1}{8}$. By the sine area formula, the area of $\triangle ABC$ is $\frac{1}{2}ac\sin(\theta)$, the area of $\triangle ABD$ is $\frac{1}{2}cd\sin(\frac{\theta}{2})$, and the area of $\triangle BCD$ is $\frac{1}{2}ad\sin(\frac{\theta}{2})$. Since $[ABC] = [ABD] + [BCD]$, we obtain the equality $\frac{1}{2}ac\sin(\theta) = \frac{1}{2}cd\sin(\frac{\theta}{2}) + \frac{1}{2}ad\sin(\frac{\theta}{2})$, so upon rearranging we see that $d = \frac{cd}{c+d} \cdot \frac{\sin(\theta)}{\sin(\theta/2)}$. The given information then implies $\frac{cd}{c+d} = \frac{1}{1 + \frac{1}{d}} = 8$ and $\frac{\sin(\theta)}{\sin(\theta/2)} = 2\cos(\theta/2) = 2\sqrt{1 + \cos^2(\theta)/2} = 2 \cdot \sqrt{\frac{9}{16}} = \frac{3}{2}$, and therefore $d = 8 \cdot \frac{3}{2} = [12]$.

**Remark:** There are infinitely many different triangles satisfying the two conditions in the problem statement. One possibility is the isosceles triangle with $AB = BC = 16$: then the angle bisector becomes an altitude, and the length of side $AC$ may be calculated via the Law of Cosines: $AC^2 = 16^2 + 16^2 - 2 \cdot 16 \cdot 16 \cdot \frac{1}{8} = 448$ so that $AC = 8\sqrt{7}$. In this triangle, we have $d^2 = 16^2 - (4\sqrt{7})^2 = 144$ so that $d = 12$ (of course).

6. Find an ordered triple $(a, b, c)$ of positive integers with $a < b < c$ such that

$$\sqrt{\frac{1}{a}} + \sqrt{\frac{1}{a+1}} + \sqrt{\frac{1}{b}} + \sqrt{\frac{1}{b+1}} + \sqrt{\frac{1}{c}} + \sqrt{\frac{1}{c+1}} = 1.$$

**Answer:** $(a, b, c) = (8, 24, 48)$.

**Solution:** By rationalizing the denominator, observe that $\frac{1}{\sqrt{a} + \sqrt{a+1}} = \sqrt{a+1} - \sqrt{a}$, and so we seek expressions where $\sqrt{a+1} - \sqrt{a}$ will simplify. If this simplification is to be into the form $\sqrt{b} - \sqrt{c}$, then setting $\sqrt{a+1} - \sqrt{a} = \sqrt{b} - \sqrt{c}$ and squaring yields $\sqrt{a+1} - \sqrt{a} = (b + c) - 2\sqrt{bc}$, from which we want $4bc = a$ and $(b + c)^2 = a + 1$. In particular, we want $a$ to be even and $a + 1$ to be a perfect square (hence, an odd perfect square). But now notice $3 - \sqrt{8} = (\sqrt{2} - 1)^2$, that $5 - \sqrt{24} = (\sqrt{3} - \sqrt{2})^2$, and that $7 - \sqrt{48} = (2 - \sqrt{3})^2$. (Each of these is a special case of the general relation $\sqrt{4k^2 + 4k + 1} - \sqrt{4k^2 - 4k} = k + 1 - \sqrt{k}$.) Thus, we can write $\frac{1}{\sqrt{8} + \sqrt{9}} = (\sqrt{2} - 1)^2$, $\frac{1}{\sqrt{24} + \sqrt{25}} = (\sqrt{3} - \sqrt{2})^2$, and $\frac{1}{\sqrt{48} + \sqrt{49}} = (\sqrt{3} - \sqrt{2})^2$, and therefore $\sqrt{\frac{1}{8}} + \sqrt{\frac{1}{9}} + \sqrt{\frac{1}{24}} + \sqrt{\frac{1}{25}} + \sqrt{\frac{1}{48}} + \sqrt{\frac{1}{49}} = (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) = 1$. Thus, $(8, 24, 48)$ is one possible triple.

**Remark:** The solution above does not prove that $(8, 24, 48)$ is the only possible triple. (The authors expect this to be the case, and have verified there is no other triple with $c < 100$.) We also note that $(8, 8, 288)$ is another triple satisfying the second condition, although two of the numbers are equal.

**Note:** For full credit, it was necessary to give a justification why a particular triple yields an answer to the problem. It would not be enough to say (for example) that numerically evaluating the expression for $(a, b, c) = (8, 24, 48)$ yields a result that appears to be equal to 1. There are numerous other triples for which the expression is very close to 1 but not exactly equal, such as $(a, b, c) = (2, 7, 41244980)$, where the expression is equal to $1.00000000008373$ to 15 decimal places.