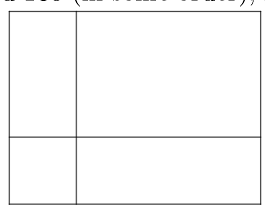
# Vermont Mathematics Talent Search, Solutions to Test 1, 2018-2019

Test and Solutions by Kiran MacCormick and Evan Dummit

November 6, 2018

1. A large rectangle is divided into three smaller rectangles and a square as shown in the diagram. If the three nonsquare rectangles have areas 96, 120, and 180 (in some order), find all possible side lengths for the square.



**Answer:** 8, 12, 15.

**Solution:** Suppose the square has side length $s$. If the two segments on the interior are divided into pieces of lengths $a, s$ and $b, s$ respectively, then the areas of the four smaller rectangles (including the square) are $ab$, $as$, $bs$, and $s^2$. The given information then says that $ab$, $as$, and $bs$ are equal to 96, 120, and 180 in some order. Noting that $s^2 = \dfrac{as \cdot bs}{ab}$, we see that of the following three possibilities must hold: $s^2 = \dfrac{96 \cdot 120}{180} = 64$ so that $s = 8$, or $s^2 = \dfrac{96 \cdot 180}{120} = 144$ so that $s = 12$, or $s^2 = \dfrac{120 \cdot 180}{96} = 225$ so that $s = 15$. It is not hard to verify that each of these three possibilities can actually occur: $(a, b, s) = (8, 12, 15)$ yields $(ab, as, bs, s^2) = (96, 120, 180, 225)$, while $(a, b, s) = (8, 15, 12)$ yields $(ab, as, bs, s^2) = (120, 96, 180, 225)$, and $(a, b, s) = (12, 15, 8)$ yields $(ab, as, bs, s^2) = (180, 96, 120, 64)$. Therefore, the possible side lengths are $s = \boxed{8,\ 12,\ 15}$.

2. Kat writes down the units digits of the numbers $1^1$, $2^2$, $3^3$, ... , $2016^{2016}$, $2017^{2017}$, $2018^{2018}$, obtaining a sequence 1, 4, 7, ... , 6, 7, 4. What is the sum of the numbers in Kat's sequence?

**Answer:** 9485.

**Solution:** Using a computer or otherwise, the first 20 terms of the sequence are {1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0}, and the next 20 terms are also {1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0}. This suggests that the sequence repeats every 20 terms. In fact, this is true: for any digit $d$, first observe that $d^5 \equiv d \pmod{10}$, either by a direct check or by combining the two observations that $d^5 \equiv d \pmod 2$ and $d^5 \equiv d \pmod 5$. Then we see that $(d+20)^{d+20} \equiv d^{d+20} \equiv d^d \pmod{10}$, by applying the relation $d^5 \equiv d \pmod{10}$ five times. Hence, the sequence repeats every 20 terms as claimed.

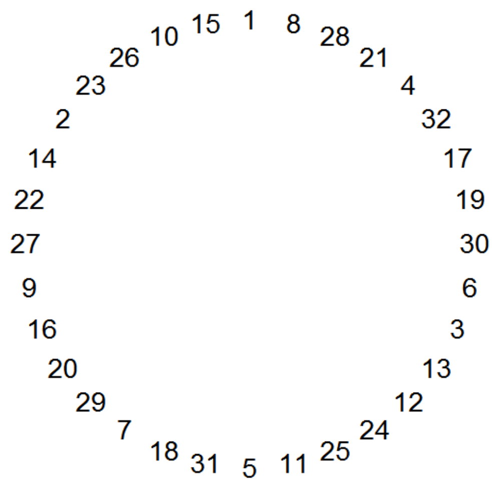
Then the terms 1 through 2000 consist of 100 copies of {1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0}, with total sum $100 \cdot 94$, along with the first 18 terms again, which have sum 85. The total sum of all the terms is therefore $100 \cdot 94 + 85 = \boxed{9485}$.

3. Prove that the integers 1 through 32 inclusive can be arranged around a circle so that the sum of each pair of adjacent integers is a perfect square, but that the integers 1 through $n$ inclusive cannot be so arranged for any integer $n$ with $2 \le n \le 30$.

**Solution:** Call two integers "connected" if their sum is a square. In order to arrange the integers in the required way, each integer in $\{1, 2, \ldots, n\}$ must be connected to at least two other integers, since each integer must have two neighbors on the circle. If $2 \le n \le 6$, then 2 is not connected to any other integer, while if $7 \le n \le 13$, then 2 is connected only to 7. If $14 \le n \le 15$, then 9 is connected only to 7. If

$16 \leq n \leq 19$, then 16 is connected only to 9. Finally, if $20 \leq n \leq 30$, then 18 is connected only to 7. This eliminates any possible $n$ with $2 \leq n \leq 30$.

For $n = 32$, there are several numbers that are only connected to two others (such as 32, which is only connected to 4 and 17). These yield the pieces 4-32-17, 5-31-18, 6-30-19, 7-29-20, 8-28-21, 9-27-22, 8-26-23, 9-25-24, 7-18-31, and 9-16-20. Several of these pieces must then join together since they share common numbers, and these connections then exclude various other possible adjacencies (e.g., 9-27-22 must join to 9-16-20 to form 22-27-9-16-20, and then this excludes the possibility that 9 can be adjacent to 7). Repeating logic of this nature, and avoiding loops, eventually yields a single arrangement (unique up to reversal and reordering):




**Remark:** When $n = 31$, each integer is connected to at least two others, but it is still not possible to create a satisfactory arrangement. (The only proof of this fact that the problem authors could find involved substantial case analysis similar to the construction for $n = 32$, and for this reason the case $n = 31$ was excluded from the problem.)

4. If $f(n) = \dfrac{n-3}{\sqrt{2n-2}-2}$, what is the smallest integer $n > 3$ such that $f(f(f(f(n))))$ is an integer?

**Answer:** $2^{31} + 1 = 2147483649$.

**Solution 1:** Observe that for $n \neq 3$, we have

$$f(n) = \frac{n-3}{\sqrt{2n-2}-2} \cdot \frac{\sqrt{2n-2}+2}{\sqrt{2n-2}+2} = \frac{(n-3)\cdot(\sqrt{2n-2}+2)}{2n-6} = \frac{\sqrt{2n-2}+2}{2} = \sqrt{\frac{n-1}{2}}+1.$$

Then $f(f(n)) = \sqrt[4]{\dfrac{n-1}{8}}+1$, $f(f(f(n))) = \sqrt[4]{\dfrac{n-1}{128}}+1$, and $f(f(f(f(n)))) = \sqrt[16]{\dfrac{n-1}{8^5}}+1$.
Solving $f(f(f(f(n)))) = k$ yields $n = 8^5(k-1)^{16}+1$.
If $k = 1$ then we get $n = 1$, which does not have $n > 1$.
If $n = 2$ then we get $n = 8^5 + 1 = 32769$, but this does not work either because $f(f(f(n))) = 3$ and so $f(f(f(f(n))))$ is undefined.
If $k = 3$ then we get $n = 8^5 \cdot 2^{16} + 1 = \boxed{2147483649}$. This value does work, as $f(n) = 32769$, $f(f(n)) = 129$, $f(f(f(n))) = 9$, and $f(f(f(f(n)))) = 3$ are all defined.

**Solution 2:** Since $f(x)$ is an increasing function when $x > 3$ (as can be shown by a careful algebraic estimate, or using calculus) and $f(x)$ =1 has only the solution $x = 1$, the smallest possible candidate for $n$ would have $f(f(f(f(n)))) \geq 2$. However, solving $f(f(f(f(n)))) = 2$ yields an invalid solution for $n$ (solving the equation algebraically yields $f(f(f(n))) = 3$ but this value is not in the domain of $f$), and so we must in fact have $f(f(f(f(n)))) \geq 3$. Solving $f(x) = 3$ yields $x = 9$, so we must have $f(f(f(n))) \geq 9$. Solving $f(x) = 9$ yields $x = 129$, so we must have $f(f(n)) \geq 129$. Solving $f(x) = 129$ yields $x = 32769$, so we must have $f(n) \geq 32769$. Finally, solving $f(n) = 32769$ yields $n = \boxed{2147483649}$, and the chain of inequalities above shows that no smaller value of $n$ can work.

5. Triangle $ABC$ has $\frac{1}{AB} + \frac{1}{BC} = \cos(B) = \frac{1}{8}$. Find the length of the bisector of angle $B$.

**Answer:** 12.

**Solution:** Let the bisector of angle $B$ intersect $AC$ at $D$, and take $AB = c$, $BC = a$, $BD = d$, and $m\angle ABC = \theta$: then we are given $\frac{1}{c} + \frac{1}{d} = \cos(\theta) = \frac{1}{8}$. By the sine area formula, the area of $\triangle ABC$ is $\frac{1}{2}ac\sin(\theta)$, the area of $\triangle ABD$ is $\frac{1}{2}cd\ \sin(\frac{\theta}{2})$, and the area of $\triangle BCD$ is $\frac{1}{2}ad\ \sin(\frac{\theta}{2})$. Since $[ABC] = [ABD] + [BCD]$, we obtain the equality $\frac{1}{2}ac\ \sin(\theta) = \frac{1}{2}cd\ \sin(\frac{\theta}{2}) + \frac{1}{2}ad\ \sin(\frac{\theta}{2})$, so upon rearranging we see that $d = \frac{cd}{c+d} \cdot \frac{\sin(\theta)}{\sin(\theta/2)}$. The given information then implies $\frac{cd}{c+d} = \frac{1}{\frac{1}{c}+\frac{1}{d}} = 8$ and $\frac{\sin(\theta)}{\sin(\theta/2)} = 2\cos(\theta/2) = 2\sqrt{\frac{1+\cos\theta}{2}} = 2 \cdot \sqrt{\frac{9}{16}} = \frac{3}{2}$, and therefore $d = 8 \cdot \frac{3}{2} = \boxed{12}$.

**Remark:** There are infinitely many different triangles satisfying the two conditions in the problem statement. One possibility is the isosceles triangle with $AB = BC = 16$: then the angle bisector becomes an altitude, and the length of side AC may be calculated via the Law of Cosines: $AC^2 = 16^2 + 16^2 - 2 \cdot 16 \cdot 16 \cdot \frac{1}{8} = 448$ so that $AC = 8\sqrt{7}$. In this triangle, we have $d^2 = 16^2 - (4\sqrt{7})^2 = 144$ so that $d = 12$ (of course).

6. Find an ordered triple $(a, b, c)$ of positive integers with $a < b < c$ such that

$$\sqrt{\frac{1}{\sqrt{a}+\sqrt{a+1}}} + \sqrt{\frac{1}{\sqrt{b}+\sqrt{b+1}}} + \sqrt{\frac{1}{\sqrt{c}+\sqrt{c+1}}} = 1.$$

**Answer:** $(a, b, c) = (8, 24, 48)$.

**Solution:** By rationalizing the denominator, observe that $\frac{1}{\sqrt{a}+\sqrt{a+1}} = \sqrt{a+1} - \sqrt{a}$, and so we seek expressions where $\sqrt{\sqrt{a+1}-\sqrt{a}}$ will simplify. If this simplification is to be into the form $\sqrt{b} - \sqrt{c}$, then setting $\sqrt{\sqrt{a+1}-\sqrt{a}} = \sqrt{b} - \sqrt{c}$ and squaring yields $\sqrt{a+1} - \sqrt{a} = (b+c) - 2\sqrt{bc}$, from which we want $4bc = a$ and $(b+c)^2 = a+1$. In particular, we want $a$ to be even and $a+1$ to be a perfect square (hence, an odd perfect square). But now notice $3 - \sqrt{8} = (\sqrt{2}-1)^2$, that $5 - \sqrt{24} = (\sqrt{3}-\sqrt{2})^2$, and that $7 - \sqrt{48} = (2-\sqrt{3})^2$. (Each of these is a special case of the general relation $\sqrt{\sqrt{4k^2+4k+1}-\sqrt{4k^2+4k}} = \sqrt{k+1} - \sqrt{k}$.) Thus, we can write $\frac{1}{\sqrt{8}+\sqrt{9}} = (\sqrt{2}-1)^2$, $\frac{1}{\sqrt{24}+\sqrt{25}} = (\sqrt{3}-\sqrt{2})^2$, and $\frac{1}{\sqrt{48}+\sqrt{49}} = (2-\sqrt{3})^2$, and therefore $\sqrt{\frac{1}{\sqrt{8}+\sqrt{9}}} + \sqrt{\frac{1}{\sqrt{24}+\sqrt{25}}} + \sqrt{\frac{1}{\sqrt{48}+\sqrt{49}}} = (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (2-\sqrt{3}) = 1$. Thus, $\boxed{(8, 24, 48)}$ is one possible triple.

**Remark:** The solution above does not prove that $(8, 24, 48)$ is the only possible triple. (The authors expect this to be the case, and have verified there is no other triple with $c < 100$.) We also note that $(8, 8, 288)$ is another triple satisfying the second condition, although two of the numbers are equal.

**Note:** For full credit, it was necessary to give a justification why a particular triple yields an answer to the problem. It would not be enough to say (for example) that numerically evaluating the expression for $(a, b, c) = (8, 24, 48)$ yields a result that appears to be equal to 1. There are numerous other triples for which the expression is very close to 1 but not exactly equal, such as $(a, b, c) = (2, 7, 41244980)$, where the expression is equal to 1.00000 00000 08373 to 15 decimal places.