

Vermont Mathematics Talent Search, Solutions to Test 2, 2019-2020

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1. A cryptarithm is a correct arithmetic equation encoded using letters such that each appearance of a letter has the same value (a digit 0-9) and such that different letters have different values. (Note that numbers are not allowed to have leading zeroes.) The cryptarithm given below has seven possible solutions. If $O = 7$ and $R = 5$, what is the value of the word OUTWEIGH?

$$\begin{array}{r}
 \text{TWO} \\
 \text{TWO} \\
 + \text{FOUR} \\
 \text{-----} \\
 \text{EIGHT}
 \end{array}$$

Answer: 73941062.

Solution: From the given information $O = 7$ and $R = 5$, adding up the units column shows that $T = 9$. Next, note that the sum $\text{TWO} + \text{TWO} + \text{FOUR}$ is less than $9799 + 2 \cdot 997 = 11793$, so since numbers cannot start with 0, we must have $E = 1$. Then since $I \neq E$ and the value of EIGHT is between 10000 and 11999, the only possibility is to have $I = 0$. Also, if $F \neq 8$, then since 9 and 7 are already used, we would have $F \leq 6$, but then the sum $\text{TWO} + \text{TWO} + \text{FOUR}$ would be less than $6795 + 2 \cdot 987 < 10000$, which is impossible. Therefore we must have $F = 8$. So far, we have the following:

$$\begin{array}{r}
 9W7 \\
 9W7 \\
 + 87U5 \\
 \text{-----} \\
 10GH9
 \end{array}$$

Now looking at the hundreds column, since $9 + 9 + 7 = 25$, we see that G is at least 5. Since 5 and 7 are already used and there can be a carry of at most 2 into that column, the only possibility is $G = 6$.

$$\begin{array}{r}
 9W7 \\
 9W7 \\
 + 87U5 \\
 \text{-----} \\
 106H9
 \end{array}$$

The problem then reduces to having $2W + U + 1 = H + 10$, so that $2W + U - H = 9$, where W , U , and H are 2, 3, and 4 in some order. Since the largest possible value of $2W + U - H$ is $2(4) + 3 - 2 = 9$, we must have $W = 4$, $U = 3$, and $H = 2$. Then the word OUTWEIGH is 73941062.

2. Twenty couples attend a party. During the party, each pair of male partygoers chats twice, and each pair of female partygoers chats once. If a total of 822 chats occurred during the party, what is the greatest possible number of male-female couples who could have attended the party in total?

Answer: 12.

Solution: Suppose there are $40 - n$ male partygoers and n female partygoers. Then there are $\binom{40 - n}{2} = \frac{(40 - n)(39 - n)}{2}$ pairs of male partygoers and $\binom{n}{2} = \frac{n(n - 1)}{2}$ pairs of female partygoers, so the total

number of chats is $2 \cdot \frac{(40-n)(39-n)}{2} + 1 \cdot \frac{n(n-1)}{2} = (40-n)(39-n) + \frac{1}{2}n(n-1) = \frac{3}{2}n^2 - \frac{159}{2}n + 1560$. We therefore obtain the equation $\frac{3}{2}n^2 - \frac{159}{2}n + 1560 = 822$. Rearranging and factoring yields $\frac{3}{2}(n-12)(n-41) = 0$, so that $n = 12$ or $n = 41$. However, we cannot have $n = 41$ since this would yield a negative number of female partygoers, so we must have $n = 12$. Then there are 12 female partygoers and 28 male partygoers, so the greatest possible number of male-female couples is $\boxed{12}$.

3. Find all ordered pairs (x, y) of nonzero rational numbers satisfying the equations $x^{y-3x} = y^5$ and $y^{y-3x} = x^{20}$.

Answer: $(x, y) = (1, 1), (-1, 1), (5, 25),$ and $(-2, 4)$.

Solution: If $x = 1$, then the first equation yields $y^5 = 1$ so that $y = 1$ also, yielding the solution $(1, 1)$. If $x = -1$, then the first equation yields $(-1)^{y+3} = y^5$ so that $y^2 = (-1)^y$, which has only the solution $y = 1$, yielding the solution $(-1, 1)$.

Now assume $|x| \neq 1$ and observe that $x^{100} = (y^{y-3x})^5 = y^{5(y-3x)} = (y^5)^{y-3x} = x^{(y-3x)^2}$. Since $|x| \neq 1$, we must have $100 = (y-3x)^2$, and so $y-3x = 10$ or -10 .

If $y-3x = 10$, then the equations become $x^{10} = y^5$ and $y^{10} = x^{20}$. Taking the 5th root of the first equation produces $y = x^2$, which is also consistent with the second equation. Then the condition $y-3x = 10$ yields $x^2 - 3x = 10$, which upon rearranging and factoring yields $(x-5)(x+2) = 0$. Then $x = 5$ yields $y = 25$, producing the solution $(5, 25)$, and $x = -2$ yields $y = 4$, producing the solution $(-2, 4)$.

If $y-3x = -10$, then the equations become $x^{-10} = y^5$ and $y^{-10} = x^{20}$. Taking the 5th root of the first equation produces $y = x^{-2}$, which is also consistent with the second equation. Then the condition $y-3x = -10$ yields $x^{-2} - 3x = -10$, which upon multiplying by x^2 and rearranging yields the equation $3x^3 - 10x^2 - 1 = 0$. By the rational root test, the only possible rational roots of this polynomial are $x = \pm 1$ and $x = \pm 1/3$, and none of these are roots. Hence there are no solutions in this case.

We conclude that there are four rational solutions: $(x, y) = \boxed{(1, 1), (-1, 1), (5, 25), (-2, 4)}$.

4. Suppose that $A, B, C,$ and D are distinct lattice points in the plane such that all six lengths $AB, AC, AD, BC, BD,$ and CD are integers. Prove that the product of these six lengths is divisible by 12.

Solution: We will show that at least two of the distances must be even and at least one of the distances must be a multiple of 3.

We first observe that any lattice triangle with integer side lengths must have at least one side of even length. To see this, call a point “odd-sum” if the sum of its coordinates is odd, and “even-sum” if the sum of its coordinates is even. Then the distance between any two even-sum points or two odd-sum points is even (because their coordinates either both have the same parity or both have the opposite parity, and in either case the sum of the squares of the differences of the coordinates is even), while the distance between an odd-sum point and an even-sum point is odd (because one pair of coordinates will have the same parity while the other pair will have the opposite parity, so the sum of the squares of the differences of the coordinates is odd). By the pigeonhole principle, any lattice triangle must have two points of the same type, so it has an even side length.

Then we see $\triangle ABC$ has at least one side of even length. If it has two sides of even length we are done, so suppose without loss of generality that AB has even length and AC, BC have odd length. Then $\triangle BCD$ must also have a side of even length, which cannot be BC since its length was assumed to be odd. Hence at least one of BD and CD has even length, so along with AB we conclude there are at least 2 segments of even length.

Now we claim at least one of the distances must also be divisible by 3. To see this, first observe that if p and q are integers such that $\sqrt{p^2 + q^2}$ is also an integer, then at least one of p and q must be divisible by 3. This follows by observing that if neither p nor q is divisible by 3, then p^2 and q^2 are both congruent to 1 modulo 3, so $p^2 + q^2$ would be congruent to 2 modulo 3, but 2 is not a square modulo 3.

Now apply the observations to the differences of the coordinates of A, B, C, D considered modulo 3. If we translate all points so that $A = (0, 0) \pmod 3$, then the observation above shows that each of B, C, D must have at least one coordinate equal to 0 (mod 3). If any point has both coordinates congruent to 0 mod 3, then its distance to A is a multiple of 3, so suppose this is not the case. We cannot have a point with x -coordinate 0 mod 3 and y -coordinate not zero mod 3, and another point with x -coordinate not

zero mod 3 and y -coordinate 0 mod 3, because then neither of the differences of the coordinates would be a multiple of 3. Hence the only possibilities are either to have all three points have x -coordinate 0 mod 3 (then two points necessarily have equal y -coordinates mod 3, since there are only two nonzero possibilities, and then the distance between these points is a multiple of 3), or all three points have y -coordinate 0 mod 3 (in which case the same logic applies).

Hence in all cases, there exists at least one length that is divisible by 3.

We have shown that at least two lengths are divisible by 2 and at least one length is divisible by 3, so the product of the lengths is divisible by 12, as required.

Remark: The choice $A = (0, 0)$, $B = (1, 0)$, $C = (2, 0)$, $D = (3, 0)$ shows that the product of the six lengths is not necessarily divisible by any integer larger than 12. The question becomes more interesting if it is assumed that no three of the points are collinear: it is easy to construct examples where the points form a rectangle or a rhombus using 2 or 4 copies of a Pythagorean right triangle, but because the product of the side lengths of a Pythagorean right triangle is always divisible by 12, all such examples will necessarily have the product of lengths divisible by 48.

Another construction of such quadrilaterals can be given using the unit circle: if α and β are angles in a Pythagorean right triangle (i.e., with $\sin \alpha, \cos \alpha, \sin \beta, \cos \beta$ rational), then it is not hard to check that the points $(0, 0)$, $(1, 0)$, $(\cos 2\alpha, \sin 2\alpha)$, $(\cos 2\beta, \sin 2\beta)$ are all a rational distance from one another, and by clearing denominators, one obtains a quadrilateral with all integer side lengths. One such quadrilateral is formed by $(0, 0)$, $(25, 0)$, $(\pm 7, 24)$ with distances 14, 25, 25, 25, 30, 40; this example shows that the product of the lengths need not be divisible by 9. However, an example of this type will always have the product of side lengths divisible by 8, since if the radius of the circle is odd then the remaining three segments will be even, while if the radius of the circle is even then the three radii will be even.

5. For a positive integer n , let $f(n) = \sum_{d|n} i^d$, where $i = \sqrt{-1}$. Thus, for example, $f(10) = i^1 + i^2 + i^5 + i^{10} = -2 + 2i$.

Find the smallest positive integer n for which $f(n) = 20 + 20i$.

Answer: 1105000.

Solution: Note that $f(n) = (a - b) + (c - d)i$ where a is the number of divisors of n divisible by 4, b is the number of divisors of n congruent to 2 modulo 4, c is the number of divisors of n congruent to 1 modulo 4, and d is the number of divisors congruent to 3 modulo 4.

Write $n = 2^M p_1^{A_1} \cdots p_k^{A_k} q_1^{A_1} \cdots q_l^{A_l}$ where the p_i are distinct primes congruent to 1 modulo 4, and the q_i are distinct primes congruent to 3 modulo 4.

If $d = 2^m p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_l^{b_l}$, then d is divisible by 4 precisely when $m = 2$, d is congruent to 2 modulo 4 precisely when $m = 1$, and d is odd precisely when $m = 0$. When d is odd, we will have $d \equiv 1 \pmod 4$ precisely when $b_1 + \cdots + b_l$ is odd and $d \equiv 3 \pmod 4$ precisely when $b_1 + \cdots + b_l$ is even.

Thus, we see that the number of divisors of n is divisible by 4 is equal to $(M-1) \cdot (A_1+1) \cdots (B_l+1)$ if $M \geq 2$, and is 0 otherwise, while the number of divisors congruent to 2 modulo 4 is equal to $(A_1+1) \cdots (B_l+1)$ if $M \geq 1$, and is 0 otherwise.

If any of the b_i is odd, then there are equal numbers of divisors of n congruent to 1 modulo 4 and congruent to 3 modulo 4: there are an equal number of divisors whose power of q_i is equal to each of $0, 1, 2, \dots, q_i$ respectively, and half of them are congruent to 3 modulo 4. (Explicitly, for any odd number r , one of r and rq_i is congruent to 1 modulo 4 and the other is congruent to 3 modulo 4.)

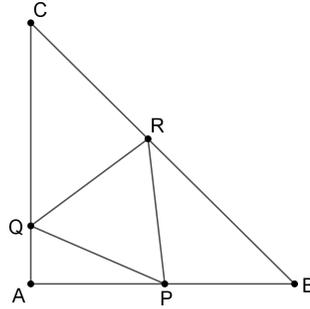
If all of the b_i are even, then we may pair up the odd divisors of n as above, but the divisors with all powers of q_i equal to 0 are unpaired, so there are $(A_1+1)(A_2+1) \cdots (A_k+1)$ additional divisors.

Thus, if we set $A = (A_1+1)(A_2+1) \cdots (A_k+1)$ and $B = (B_1+1) \cdots (B_l+1)$ we see that $f(n) = \begin{cases} 0 & \text{if } M = 0 \\ (M-2)AB & \text{if } M \geq 1 \end{cases} + \begin{cases} 0 & \text{if any } q_i \text{ is odd} \\ Ai & \text{if all } q_i \text{ are even} \end{cases}$.

Now suppose $f(n) = 20 + 20i$. To have imaginary part $20i$, we must have all q_i even along with $A = 20$, and then to have real part 20, we must have $B = 1$ and $M = 3$. We can then tabulate the various possibilities for the prime factorization of n : there are no primes congruent to 3 modulo 4, and to get $A = 20$, we may use either $A_1 = 19$, or $A_1 = 9$ and $A_2 = 1$, or $A_1 = 4$ and $A_2 = 3$, or $A_1 = 4$ with $A_2 = A_3 = 1$. The least possible n yielding these exponents are $2^3 \cdot 5^{19}$, $2^3 \cdot 5^9 \cdot 13$, $2^3 \cdot 5^4 \cdot 13^3$, and $2^3 \cdot 5^4 \cdot 13 \cdot 17$. The smallest of these is $2^3 \cdot 5^4 \cdot 13 \cdot 17 = \boxed{1105000}$.

6. Right triangle ABC has side lengths 7 , $4\sqrt{3}$, and $\sqrt{97}$. Equilateral triangle PQR is contained in triangle ABC , and has one vertex on side AB , one vertex on side BC , and one vertex on side AC as shown below.

- (a) Find the maximum possible area of triangle PQR .
 (b) Find the minimum possible area of triangle PQR .



Answer: (a) Maximum area is $\frac{2352\sqrt{3}}{361}$, (b) Minimum area is $\frac{588\sqrt{3}}{181}$.

Solution: Embed triangle ABC in the complex plane so that A is located at the origin $0+0i$, B is located at the point $7+0i$, and C is located at the point $0+4\sqrt{3}i$. Then select P on AB with coordinates $7r$, Q on AC with coordinates $(4\sqrt{3}i)s$, and R on BC with coordinates $(1-t)7+t(4\sqrt{3}i)$: then the condition that PQR is contained in ABC is equivalent to the conditions that r, s, t are real numbers with $0 \leq r, s, t \leq 1$. Now observe that triangle PQR will be equilateral if and only if the complex number $Q - P$ has the same magnitude as the complex number $R - P$ (since these are the lengths of the sides PQ and PR) and make an angle of $\pi/3$ radians (since this is the angle between PQ and PR). These two conditions are together equivalent to the statement that $R - P = e^{-i\pi/3}(Q - P)$, since multiplication by $e^{-i\pi/3}$ corresponds to a clockwise rotation of $\pi/3$ radians (without changing magnitudes).

Writing this condition out explicitly yields $(7 - 7t - 7r) + (4t\sqrt{3})i = \left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \cdot [-7r + 4\sqrt{3}si]$, so upon multiplying by 2 and distributing, we obtain $(14 - 14t - 14r) + (8t\sqrt{3})i = (-7r + 12s) + (-7r\sqrt{3} + 4s\sqrt{3})i$. Equating real and imaginary parts yields the two equations $14 - 14t - 14r = -7r + 12s$ and $8t = -7r + 4s$, so that $7r + 12s + 14t = 14$ and $7r + 4s - 8t = 0$. Solving for r and s in terms of t (e.g., by subtracting the equations to eliminate r , and then solving for s in terms of t and substituting back) yields $r = \frac{19t - 7}{7}$ and $s = \frac{7 - 11t}{4}$. Since $0 \leq r, s, t \leq 1$, the condition $0 \leq r \leq 1$ requires $\frac{7}{19} \leq t \leq \frac{14}{19}$, while the condition $0 \leq s \leq 1$ requires $\frac{3}{11} \leq t \leq \frac{7}{11}$. Together these yield an allowable range of $\frac{7}{19} \leq t \leq \frac{7}{11}$.

The side length of triangle PQR is $|Q - P| = |7r + 4\sqrt{3}si| = \sqrt{49r^2 + 48s^2}$, so the area is $\frac{(49r^2 + 48s^2)\sqrt{3}}{4} = (181t^2 - 182t + 49)\sqrt{3}$ upon substituting $r = \frac{19t - 7}{7}$ and $s = \frac{7 - 11t}{4}$. Since this quadratic polynomial in t has positive leading coefficient, its minimum value occurs at the vertex with for $t = \frac{91}{181}$, which is in the allowed interval, and the minimum value (after simplification) is $\frac{588\sqrt{3}}{181}$. The maximum value of the quadratic occurs at the endpoint farther from the vertex. Since $\frac{91}{181} - \frac{7}{19} > \frac{7}{11} - \frac{91}{181}$, the maximum area occurs at $t = \frac{7}{19}$, and this area is $\frac{2352\sqrt{3}}{361}$.

Thus, for (a), the maximum area is $\boxed{\frac{2352\sqrt{3}}{361}}$ while for (b), the minimum area is $\boxed{\frac{588\sqrt{3}}{181}}$.