Vermont Mathematics Talent Search, Solutions to Test 4, 2020-2021

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. For each of the 336 ordered triples of three distinct vertices (A, B, C) of a cube, Evan calculates the angle measure $m\angle ABC$. In degrees, what is the total sum of all of the distinct angle measures on Evan's list?

   Answer: $285°$.

   Solution: There are three types of vertex triangles in a cube of side length 1: there are isosceles triangles with sides $(1, 1, \sqrt{2})$ having angle measures $45°$, $45°$, $90°$, there are right triangles with sides $(1, \sqrt{2}, \sqrt{3})$ having angle measures $\sin^{-1}(1/\sqrt{3})$, $\cos^{-1}(1/\sqrt{3})$, $90°$, and there are equilateral triangles with sides $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ having angle measures $60°$, $60°$, $60°$. Thus, there are 5 possible angle measures: $45°$, $60°$, $90°$, and the two complementary angles $\sin^{-1}(1/\sqrt{3})$ and $\cos^{-1}(1/\sqrt{3})$. The total sum of their degree measures is $45° + 60° + 90° + 90° = \boxed{285°}$.

2. The sequence $a_0, a_1, a_2, \ldots$ of integers satisfies the conditions $a_0 = 0$ and $|a_n - a_{n-1}| = n^2$ for each positive integer n. Find the minimal possible value of k such that there exists such a sequence with $a_k = 2021$.

   Answer: 21.

   Solution: It is easy to see that $a_n = 0 \pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm n^2$ for each positive integer n and any possible selection of $\pm$ signs. We must determine the minimal value of k for which this expression can equal 2021. If we let $S_k = 1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$, then we see $2021 = S - 2\sum_{n\in M} n^2$ where M is the set of squares having minus signs. In particular, S_k must be odd and at least 2021.

   Since $S_{17} = 1785$ and $S_{18} = 2109$, we must have $k \geq 18$. However, if $k = 18$ then we would have $44 = (S_{18} - 2021)/2 = \sum_{n\in M} n^2$, which requires that 44 be a sum of distinct squares. But in fact, there is no sum of distinct squares that equals 44, as we can check using some case analysis:

   - If the sum includes 36, then the remainder must have sum 8, which is not possible because $1 + 4 < 8 < 9$.
   - If the sum includes 25, then the remainder must have sum 19, which is also not possible because $1 + 4 + 9 < 19$ so 16 must be included, but this only leaves 3 which is not a sum of distinct squares.
   - Otherwise, if the sum includes neither 25 nor 36, it is at most $1 + 4 + 9 + 16 = 30$, which is too small.

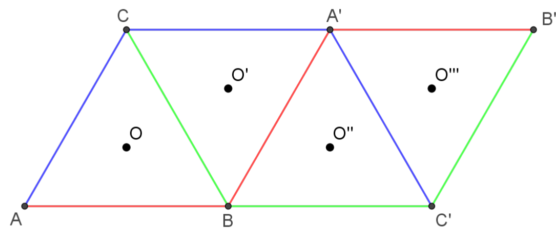
   Therefore, we must have $k \geq 19$. But $S_{19} = 2470$ and $S_{20} = 2870$ are both even, and thus 2021 cannot differ from them by an odd number. So in fact we must have $k \geq 21$. We can see, fairly straightforwardly, that $(S_{21} - 2021)/2 = 645$ is the sum of distinct squares: for example, it is $625 + 16 + 4$, and so $k = \boxed{21}$ is achievable.

3. Kiran is playing a version of capture-the-flag. His home base is located at the origin $(0, 0)$, and any point within a distance 1 of his home base is considered safe. Red flags are located at every point along the line $y = -\sqrt{3}$, blue flags are located at every point along the line $y = \sqrt{3}(x + 2)$, and green flags are located at every point along the line $y = -\sqrt{3}(x - 2)$. Kiran must start at a point that is safe, pick up a flag of each color, and then return to a point that is safe. What is the minimum total distance Kiran could travel while achieving this task?

   Answer: $2\sqrt{21} - 2$.

Solution: The points of intersection of the three lines are $A = (-3, -\sqrt{3})$, $B = (3, -\sqrt{3})$, and $C = (0, 2\sqrt{3})$, which form an equilateral triangle with $(0, 0)$ at its center. By rotating or reflecting the triangle ABC (and also Kiran's path inside the triangle), we may assume without loss of generality that he first visits side BC, then side AB, then side AC, before returning to a safe point.

When Kiran reaches side BC, reflect Kiran's path and triangle ABC across BC to create triangle $A'BC$. Then when Kiran's reflected path reaches side $A'B$, reflect $A'BC$ across $A'B$ to create triangle $A'BC'$, and finally when his doubly-reflected path reaches side $A'C'$, reflect across $A'C'$ to create triangle $A'B'C'$, as shown below.



It is straightforward to calculate that $A' = (6, 2\sqrt{3})$, $C' = (9, -\sqrt{3})$, and $B' = (12, 2\sqrt{3})$, so the center of $A'B'C'$ is $(9, \sqrt{3})$.

Kiran's reflected path is then a polygonal path that starts at some point on the circle of radius 1 centered at $(0, 0)$ then ends at some point on the circle of radius 1 centered at $(9, \sqrt{3})$. The shortest possible path of this nature follows the line joining the centers of these two circles, which has length $\sqrt{84} = 2\sqrt{21}$, and starts on the boundary of the circles. Thus, its length is $\boxed{2\sqrt{21} - 2}$.

4. Find the smallest prime number p such that a rational number of the form a/p with $0 < a < p$ contains the string "2021" somewhere in its base-10 decimal expansion. (Note that the four digits must appear in that order and cannot be separated by any other digits, but may appear anywhere in the decimal expansion.)

Answer: $p = 277$.

Solution: Suppose p has the desired property. If we multiply a/p by an appropriate power of 10 and then subtract off the integer part, we obtain a rational number b/p such that b/p has decimal expansion $0.2021\ldots$. This is equivalent to requiring that $0.2021 \leq \frac{b}{p} < 0.2022$. Multiplying by 5 and subtracting 1 yields $0.0105 \leq \frac{5b-p}{p} < 0.011$, or equivalently, $\frac{5b-p}{0.011} < p \leq \frac{5b-p}{0.0105}$. Since $5b - p$ is an integer and p is positive, we see $5b - p$ must be a positive integer. If $5b - p = 1$ then we obtain the inequality $90.90\cdots < p \leq 95.23\ldots$, but there are no prime numbers in this range. If $5b - p = 2$ then we obtain $181.81\cdots < p \leq 190.47\ldots$, but there are no primes in this range either. If $5b - p = 3$ then we obtain $272.72\cdots < p \leq 285.71\ldots$, yielding the possible candidates $p = 277$ and $p = 281$. These respectively yield $b = 56$ (which will work) and $b = 56.8$ (does not work).

Indeed, $56/277 = 0.\overline{2021660649819494584837545126353790613718411552346570397111913357400 72}$, which does indeed start with the string 2021. Therefore, $p = \boxed{277}$ is the smallest such prime.

5. Let n be an arbitrary positive integer. Prove that $1^{2021} + 2^{2021} + 3^{2021} + \cdots + n^{2021}$ is divisible by $1 + 2 + 3 + \cdots + n$.

Solution: Let $S = 1^{2021} + 2^{2021} + 3^{2021} + \cdots + n^{2021}$. Modulo n, we have $S \equiv 1^{2021} + 2^{2021} + 3^{2021} + \cdots + (n-1)^{2021}$ so, writing the sum in reverse order and using the fact that $n - k \equiv -k \bmod n$, we also have $S \equiv (-1)^{2021} + (-2)^{2021} + (-3)^{2021} + \cdots + (-(n-1))^{2021}$. Adding these two sums yields $2S \equiv [1^{2021} + (-1)^{2021}] + [2^{2021} + (-2)^{2021}] + \cdots + [(n-1)^{2021} + (-(n-1))^{2021}] \equiv 0$ since each pair of terms cancels.

In a similar way, modulo $n + 1$, we have $S \equiv 1^{2021} + 2^{2021} + 3^{2021} + \cdots + (n-1)^{2021} + n^{2021} \equiv (-1)^{2021} + (-2)^{2021} + \cdots + (-n)^{2021} \bmod (n+1)$, so $2S \equiv [1^{2021} + (-1)^{2021}] + [2^{2021} + (-2)^{2021}] + \cdots + [n^{2021} + (-n)^{2021}] \equiv 0$.

Therefore, $2S$ is zero modulo n and modulo $n+1$, so since n and $n+1$ are relatively prime, $2S$ is divisible by $n(n+1)$. This is equivalent to saying S is divisible by $\frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n$, as required.

6. A 4×4 grid of positive integers is called *divisibly correct* if each entry divides the entry directly above it and the entry directly to the left of it. How many divisibly correct grids are there whose upper-left entry is 4042 and whose bottom-right entry is 1? One such grid is shown below.

4042	4042	2021	47
4042	94	47	47
86	2	1	1
43	1	1	1

Answer: $68^3 = 314432$.

Solution: Because $4042 = 2 \cdot 43 \cdot 47$ and the only condition involves divisibility, we may consider separately which entries in the grid that are divisible by 2, 43, and 47, and the condition remains unchanged. Since the only possible entries in the grid are then 1 and p, where $p = 2$, 43, or 47, it is enough to consider the ways in which we could fill all of the entries in the grid with entries 1 or p for each of these p, and then multiply the corresponding grids entry-by-entry.

So now let p be a prime. We must compute the number of ways to fill in the grid given that the entry in the upper left is p and the entry in the lower right is 1. Each entry in the grid is then either 1 or p, so the entries in any given column are completely determined by the value of the product of those entries, which is one of $\{1, p, p^2, p^3, p^4\}$: the number of entries equal to p is determined by the exponent of p in the product, and these entries must appear above the remaining 1 entries. Furthermore, because the value in any entry is less than or equal to the entry directly to its left, the column products must be nonincreasing from left to right. Conversely, any selection of four nonincreasing terms from $\{1, p, p^2, p^3, p^4\}$, corresponding to the column products, will yield a unique divisibly-correct grid with all entries equal to 1 or p. From these tuples we must exclude the set $\{1, 1, 1, 1, 1\}$ since its upper-left entry is 1 (not p), and also the set (p^4, p^4, p^4, p^4) since its bottom-right entry is p (not 1).

By a standard stars-and-bars argument, the number of ways to choose four entries with replacement from the list of 5 terms $\{1, p, p^2, p^3, p^4\}$, where order is irrelevant, is equal to the number of ways of arranging four stars (the entries) and four bars (creating five bins representing the values of the entries), which is $\binom{8}{4} = 70$, so there are $70 - 2 = 68$ ways to organize the grid for the prime p.

We therefore obtain 68 possible grids for each of the three primes $p = 2$, $p = 43$, and $p = 47$, and since these selections are independent, the total number of grids when we combine these selections is $68^3 = \boxed{314432}$.