Vermont State Mathematics Coalition Talent Search -- January 2021
Test 3 of the 2020-2021 school year

PRINT NAME:_____ Signature:_____

School_____ Grade ـــــــــــــــــــــــــ

Current Mathematics Teacher: _____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **March 1, 2021** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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1. Kiran has 2021 different weights, and he knows that the mass of the nth weight is 2^n grams for $1 \leq n \leq 2021$. Kiran uses a balance scale that reports the signed difference in the total masses of the weights placed in its two baskets. For example, if in Basket A Kiran places the 1- and 16-gram weights, and in the second basket Kiran places the 32-gram weight, the scale would read +15 grams. Kiran places some weights in each basket, and the scale reads +4082420 grams, which Kiran notices equals 2020*2021. What is the smallest possible number of weights Kiran could have used in total?

Answer:_____

2. Find all ordered pairs (a, b) of positive integers such that $\dfrac{2021 - a}{a} \cdot \dfrac{2021 - b}{b} = 3$.

Answer:_____

3. Preston has a deck of 12 special cards: the first card says "Exactly 0 of the statements on the cards to the left of this card are true", the second card says "Exactly 1 of the statements on the cards to the left of this card are true", the third card says "Exactly 2 of the statements on the cards to the left of this card are true", and so forth, and the twelfth card says "Exactly 11 of the statements on the cards to the left of this card are true.". Preston randomly shuffles the 12 cards and then deals them in a line. What is the probability that exactly 4 cards have true statements written on them?

Answer:_____

4. A convex hexagon has 9 diagonals.
   (a) Show that it is possible for 7 of these diagonals to have the same length.
   (b) Show that it is impossible for 8 of these diagonals to have the same length.
Note: For this problem, please include your proof on separate sheets of paper.

5. A slanted cone (i.e., a cone such that the line between its vertex and the center of its base is not perpendicular to its base) has a base radius of 4 cm and a volume of 128π cm³. The largest sphere that can be inscribed in this cone has volume 36π cm³. Compute the distance from the cone's vertex to the center of its base.

Answer:_____

6. A total of 3,432 students compete in the Very Mathematical Test Season (VMTS). The VMTS has 14 true/false problems, and each student answers every problem either correctly or incorrectly. Determine the greatest possible value of N such that the following statement must be true for any possible results of the competition:
   ● There must necessarily exist a pair of 2 problems and a set of N students such that all of the N students answered both problems correctly, or all of the N students answered both problems incorrectly.

Answer:_____