Vermont Mathematics Talent Search, Solutions to Test 1, 2021-2022

Test and Solutions by Kiran MacCormick and Evan Dummit

November 11, 2021

1. In the cross-number puzzle below, each entry is a digit from 1-9. Solve the puzzle:

   Across:

   1. A 3-digit multiple of 37.
   4. An even perfect square.
   5. Twice a perfect cube.

   Down:

   1. The same number as 3-down.
   2. A multiple of 47 less than 200.
   3. The same number as 1-down.

[1]	[2]	[3]
[4]		
[5]		

   Answer:

[1] 1	[2] 1	[3] 1
[4] 4	8	4
[5] 6	8	6

   Solution: First, because 1-down and 3-down are the same number, each of the across answers is a palindrome. So for 4-across, we are searching for an even perfect square palindrome, and for 5-across, we are searching for twice a perfect cube that is a palindrome. The 3-digit even perfect squares are 100, 144, 196, 256, 324, 400, 484, 576, 676, 784, and 900, of which only 474 and 676 are palindromes. Also, the 3-digit numbers that are twice a cube are 128, 250, 432, and 686, of which only 686 is a palindrome.

   Therefore, 5-across is 686, while 4-across is either 484 or 676. Also, for 2-down, the only multiples of 47 less than 200 are 47, 94, 141, and 188. Only 188 ends in an 8 (agreeing with the 8 in 686 from 5-across), so 2-down is 188, meaning that 4-across must be 484. The only remaining digits are in the top-left and top-right, which must be equal. We are therefore looking for a multiple of 37 of the form $a1a$, and it is easy to check that the only such number is 111. The finished puzzle is thus as below:

[1] 1	[2] 1	[3] 1
[4] 4	8	4
[5] 6	8	6

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Suppose A and B are distinct, nonzero digits such that $2 \cdot \underline{ABA} + 3 \cdot \underline{BB} + 1 \cdot \underline{A} = \underline{BBB}$ (where $\underline{ABA}$ means the 3-digit integer whose digits are A, B, A, etc.). What is the value of the two-digit integer $\underline{AB}$?

   (b) Let T be the answer to part (a). What is the least positive integer that is a multiple of $T+3$ whose sum of digits is $T-3$?

   (c) Let S be the closest integer to the answer of part (b) divided by 120. A rectangle is inscribed in the circle $x^2 + y^2 = S$ and its side lengths are both integers. What is the area of this rectangle?

   Answers: (a) 27, (b) 6990, (c) 84.

   Solution (a): Since $\underline{ABA} = 100A + 10B + A$, $\underline{BB} = 11B$, and $\underline{BBB} = 111B$, the equation yields $2(101A + 10B) + 3(11B) + A = 111B$ so that $203A = 58B$ and thus $2B = 7A$. Since A and B are nonzero, we must have $A = 2$ and $B = 7$, so $\underline{AB} = \boxed{27}$.

Solution (b): Since $T = 27$, we are seeking the least positive integer that is a multiple of 30 whose sum of digits is 24. To be a multiple of 30 requires that the units digit be 0 and the sum of the digits be divisible by 3, which is guaranteed if the sum of digits is 24. So we want the smallest integer ending in a 0 whose sum of digits is 24, which is $\boxed{6990}$.

Solution (c): We have $6990/120 \approx 58.25$ so $S = 58$. We then have a rectangle inscribed in the circle $x^2 + y^2 = 58$ with integer side lengths. If the side lengths are c and d, then by the Pythagorean theorem, we have $2\sqrt{58} = \sqrt{c^2 + d^2}$ so that $c^2 + d^2 = 4 \cdot 58 = 232$. Assuming without loss of generality that $c \geq d$, then we have $\sqrt{116} \leq c \leq \sqrt{232}$, so since $\sqrt{232} \approx 15.232$ and $\sqrt{116} \approx 10.77$, testing $c = 11, 12, 13, 14, 15$ shows that only $c = 14$ yields an integral value of d, which is $d = 6$. Therefore, the area of the desired rectangle is $6 \cdot 14 = \boxed{84}$.

3. Eric has a rectangular piece of cardboard measuring x units by y units. He is interested in cutting out four squares of equal side length, one from each corner of the rectangle, to form a new piece of cardboard which he may then fold up into a rectangular box missing its top. If he cuts out a square of side length 2 units from each corner, then the resulting box has volume 72 cubic units. If instead he cuts out squares of side length 3 units, then the box has volume 42 cubic units. Find the area of the original piece of cardboard.

Answer: 104 square units.

Solution: If Eric cuts out squares of side length s, then the dimensions of the resulting box are s, $x - 2s$, and $y - 2s$, so the volume is $s(x - 2s)(y - 2s)$. We thus obtain the two equations

$$\begin{aligned} 2(x-4)(y-4) &= 72 \\ 3(x-6)(y-6) &= 42 \end{aligned}$$

or upon cancelling the leading constants,

$$\begin{aligned} (x-4)(y-4) &= 36 \\ (x-6)(y-6) &= 14. \end{aligned}$$

Subtracting the two equations and expanding yields $2x + 2y - 20 = 22$, so $x + y = 21$. Rearranging the first equation then yields $xy = 20 + 4(x + y) = 20 + 84 = \boxed{104}$.

Remark: Solving the equations for x and y will show that the dimensions of the piece are 8 units by 13 units.

4. Suppose that x is a real number such that $2^{4^{8^x}} = 4^{8^{2^x}}$. What is the value of $4^x - 2^x$?

Answer: 1/2.

Solution: Note that $4^{8^{2^x}} = 2^{2 \cdot 8^{2^x}}$, so the equality $2^{4^{8^x}} = 2^{2 \cdot 8^{2^x}}$ is equivalent to $4^{8^x} = 2 \cdot 8^{2^x}$. This equality is in turn equivalent to $2^{2 \cdot 8^x} = 2^{1 + 3 \cdot 2^x}$, and hence to $2 \cdot 8^x = 1 + 3 \cdot 2^x$. If we let $y = 2^x$, then since $8^x = y^3$, we see that y satisfies the cubic equation $2y^3 = 1 + 3y$, so that $2y^3 - 3y - 1 = 0$. Factoring yields $(y + 1)(2y^2 - 2y - 1) = 0$, so that $y = -1$ or $y = \dfrac{2 \pm \sqrt{12}}{4} = \dfrac{1 \pm \sqrt{3}}{2}$. But since $y = 2^x$ must be positive, the only possibility is $y = \dfrac{1 + \sqrt{3}}{2}$. Then $4^x - 2^x = y^2 - y = \dfrac{4 + 2\sqrt{3}}{4} - \dfrac{1 + \sqrt{3}}{2} = \boxed{\dfrac{1}{2}}$.

5. Kiran writes down the values of the products $a \times b$ where a and b are integers with $1,000,000,000,000,000,000 \leq a \leq 1,000,000,002,000,000,001$ and $1,000,000,000,000,000,000 \leq b \leq 1,000,000,002,000,000,001$, and then he erases any duplicate values. After erasing, how many different numbers does Kiran have written down?

Answer: $2 \cdot 10^{18} + 5 \cdot 10^9 + 2 = 2,000,000,005,000,000,002$.

Solution: Note that the given ranges have $10^{18} \leq a, b \leq 10^{18} + 2 \cdot 10^9 + 1$, which is to say, $(10^9)^2 \leq a, b \leq (10^9 + 1)^2$. Clearly, we have $ab = ba$ for any such pair, and also we can see that $(10^9)^2 \cdot (10^9 + 1)^2 = [10^9(10^9 + 1)]^2$. We claim that aside from these duplicates, all of the remaining pairs yield distinct products.

To see this, suppose $ab = cd$ with $(10^9)^2 \leq a, b, c, d \leq (10^9 + 1)^2$. If $a = c$ or $a = d$ then $(a, b) = (c, d)$ or (d, c). Otherwise, by swapping the labels if necessary, assume that $a < c, d$. If we let $p = \gcd(a, c)$ so that $a = pq$ and $c = pr$, then q, r are relatively prime and $qb = rd$, so q divides d and r divides b, and in fact

$d/q = b/r$. If we let $d/q = b/r = s$, then $a = pq$, $b = rs$, $c = pr$, and $d = qs$ for some integers p, q, r, s. Because $c, d > a$, we must have $r > q$ and $s > p$, and so $b = rs \geq (p+1)(q+1) = pq+p+q+1$. But since $p + q \geq 2\sqrt{pq}$ by the arithmetic-geometric mean inequality, we see $b \geq pq + 2\sqrt{pq} + 1 = a + 2\sqrt{a} + 1 \geq 10^{18} + 2\sqrt{10^{18}} + 1 = (10^9 + 1)^2$, and equality can hold only when $a = 10^{18}$ and $b = (10^9 + 1)^2$, and $p = q$ then requires $c = d$ so that $c = d = 10^9(10^9 + 1)$. So in fact the only case with $a < c, d$ where we have $ab = cd$ is the one we identified above.

Finally, we must count the individual products themselves. There are $N = 2{,}000{,}000{,}002 = 2 \cdot 10^9 + 2$ choices for a and the same number for b. Of these, N have $a = b$ and yield distinct products, and the other $N^2 - N$ are grouped into pairs having the same product, for a total of $(N^2 - N)/2$ different results. We must then subtract 1 to account for the fact that the pair $(10^{18}, 10^{18}+2\cdot 10^9+1)$ has the same product as $(10^{18}+10^9, 10^{18}+10^9)$. In total, there are $N + \dfrac{N^2 - N}{2} - 1 = (2\cdot 10^9+2) + \dfrac{(2 \cdot 10^9 + 2)(2 \cdot 10^9 + 1)}{2} - 1 = 2\cdot 10^9 + 1 + (10^9+1)(2\cdot 10^9+1) = 2\cdot 10^{18} + 5\cdot 10^9 + 2 = \boxed{2{,}000{,}000{,}005{,}000{,}000{,}002}$ different products on Kiran's list.

6. A positive integer is called *power-different* if it can be written as the difference between a power of 2 and a power of 3. For example, $1 = 3 - 2$, $2 = 3 - 1$, and $15 = 16 - 1$ are all power-different. Determine, with proof, the least prime number that is not power-different.

Answer: 41.

Solution 1: First, we show that all of the primes less than 41 are power-different:

$2 = 3 - 1$	$3 = 4 - 1$	$5 = 8 - 3$	$7 = 8 - 1$	$11 = 27 - 16$	$13 = 16 - 3$
$17 = 81 - 64$	$19 = 27 - 8$	$23 = 27 - 4$	$29 = 32 - 3$	$31 = 32 - 1$	$37 = 64 - 27$

It remains to prove that 41 is not power-different, which requires showing that the two Diophantine equations $41 = 2^a - 3^b$ and $41 = 2^c - 3^d$ have no solutions.

First suppose $41 = 2^a - 3^b$. Then $2^a > 41$ so $a \geq 6$. Now reduce both sides of $41 = 2^a - 3^b$ modulo 16: since $2^a = 0 \pmod{16}$, we must have $3^b \equiv -41 \equiv -7 \pmod{16}$. But the powers of 3 modulo 16 are $1, 3, 9, 11$, so there is no solution to $3^b \equiv -7 \pmod{16}$ and therefore no solution to $41 = 2^a - 3^b$.

Now suppose $41 = 3^c - 2^d$. Then $3^c > 41$ so $c \geq 4$, meaning also that $2^d \geq 3^4 - 41 = 40$ so $d \geq 6$. Now reduce both sides of $41 = 3^c - 2^d$ modulo 4: since $2^d \equiv 0 \pmod 4$, we must have $3^c \equiv 41 \equiv 1 \pmod 4$, which requires c to be even, say, $c = 2p$. Also, reducing both sides of $41 = 3^c - 2^d$ modulo 9 and noting that $3^c \equiv 0 \pmod 9$ yields $2^d \equiv -41 \equiv 4 \pmod 9$. Since the powers of 2 modulo 9 repeat with period 6 (they are $1, 2, 4, 8, 7, 5$), having $2^d \equiv 4 \pmod 9$ is equivalent to requiring $d \equiv 2 \pmod 6$, so in particular d is even, say, $d = 2q$.

Now, finally, we have $41 = 3^{2p} - 2^{2q} = (3^p - 2^q)(3^p + 2^q)$, and so since 41 is prime, the only possible factorization would have $3^p - 2^q = 1$ and $3^p + 2^q = 41$. But this would mean $3^p = 21$ and $2^q = 20$, which are both impossible. Therefore, there is no solution to $41 = 3^c - 2^d$ either, so we are done.

Solution 2: We give another argument to show that $41 = 3^c - 2^d$ has no solutions. As in Solution 1, reducing modulo 4 shows that c is even, and reducing modulo 9 shows that d is even. But then both 3^c and 2^d are congruent to $\pm 1 \pmod 5$, which is a contradiction because their difference then cannot be congruent to 1 modulo 5.