Vermont Mathematics Talent Search, Solutions to Test 2, 2021-2022

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1. This is a relay problem. The answer to each part will be used in the next part.

   (a) A positive integer is called *anti-palindromic* if it is not a palindrome and cannot be written as the sum of two palindromes. How many two-digit anti-palindromic integers are there?

   (b) Let T be the answer to part (a). The positive integer n has exactly T positive factors $1 = d_1 < d_2 < \cdots < d_T = n$. Given that $d_4 = 9$ and $d_7 = 2d_6 + 3d_2$, find the value of n.

   (c) Let S be the answer to part (b). A rectangular box is inscribed in a sphere. If the surface area of the box is S square meters and the sum of the twelve edge lengths of the box is 132 meters, what is the volume of the sphere in cubic meters?

   Answers: (a) 8 (b) 189 (c) 4500π cubic meters

   Solution (a): For a two-digit integer, the possible palindromes to include in a sum are the single digits (0, 1, ... , 9) along with the multiples of 11 (11, 22, 33, ... , 99). So, any number that leaves a remainder of 0, 1, ... , 9 when divided by 11 is not anti-palindromic, since it can be written as the remainder plus an appropriate multiple of 11: for example, $83 = 4 + 77$. Thus, the only possible anti-palindromic integers are those leaving a remainder of 10 when divided by 11: these are 10, 21, 32, 43, ... , 98. Of these 9 integers, we can write $10 = 9 + 1$ so it is not anti-palindromic, but the other $\boxed{8}$ integers are anti-palindromic.

   Solution (b): Since $d_4 = 9$ is a factor of n, we see that 1 and 3 are also factors of n, so this means n has exactly one additional factor that is less than 9. This factor cannot equal 2 (since then n would also be divisible by 6), or any of 4, 6, 8 (since then n would also be divisible by 2), so it is either 5 or 7. Furthermore, since n has $T = 8$ factors in total, this means n must be $3^3 \cdot p$ where $p = 5$ or 7. But in this case, the factors of n are $\{1, 3, p, 9, 3p, 27, 9p, 27p\}$ in increasing order, so $d_7 = 2d_6 + 3d_2 = 2 \cdot 27 + 3 \cdot 3 = 63$. This means $n = 3^3 \cdot 7 = \boxed{189}$.

   Solution (c): Suppose the edge lengths of the box are a, b, c meters. Then the given information says that $4(a + b + c) = 132$ and $2(ab + ac + bc) = S = 189$. Then $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc) = 33^2 - 189 = 900$, so the radius of the sphere is $r = \frac{1}{2}\sqrt{a^2 + b^2 + c^2} = 15$ meters. The volume of the sphere is then $\frac{4}{3}\pi r^3 = \boxed{4500\pi}$ cubic meters.

2. (#2-NT) There are 12,504,636,144,000 eighteen-digit integers that use each of the digits 1, 2, 3, ... ,9 exactly twice, and the mean of these integers is M. What is the smallest of these 12,504,636,144,000 integers that is greater than M?

   Answer: 556,112,233,446,778,899

   Solution: If the integer N has digits 112233...99, then so does the integer N' obtained by swapping the 1s with 9s, 2s with 8s, 3s with 7s, and 4s with 6s. Each digit of N' is 10 minus the corresponding digit of N, so $N' = 1,111,111,111,111,111,110 - N$. These integers therefore come in pairs $(N, 1,111,111,111,111,111,110 - N)$.

   The average of each pair, and therefore of the entire list, is then $1,111,111,111,111,111,110/2 = 555,555,555,555,555,555$.

   The smallest integer exceeding the mean must start with two 5s and then a 6, and then the remaining digits must be in increasing order (to make the integer as small as possible): this yields an answer of $\boxed{556,112,233,446,778,899}$.

3. (#3-GE) An equiangular 12-gon has consecutive side lengths of $a$cm, 2012cm, 2013cm, 2014cm, ... , 2021cm, and $b$cm. Find the value of $a+b$.

Answer: 4033.

Solution: Embed the 12-gon in the complex plane so that the side of length 2012 starts at 0 and ends at 2012. Then the side of length 2013 represents the vector $2013e^{2\pi i/12} = \frac{2013\sqrt{3}}{2} + \frac{2013}{2}i$, the side of length 2014 represents the vector $2014e^{4\pi i/12} = 1007 + 1007\sqrt{3}$, ... , the side of length 2021 represents the vector $2021e^{18\pi i/12} = -2021i$, the side of length a represents the vector $ae^{20\pi i/12} = \frac{a}{2} - \frac{a\sqrt{3}}{2}i$, and the side of length b represents the vector $be^{22\pi i/12} = \frac{b\sqrt{3}}{2} - \frac{b}{2}i$.

In order for this polygon to be closed, the sum of all of these vectors must be zero. Therefore, we have $2012 + 2013e^{2\pi i/12} + 2014e^{4\pi i/12} + \cdots + 2021e^{18\pi i/12} + ae^{20\pi i/12} + be^{22\pi i/12} = 0$, which if we expand out in rectangular form yields $(-1017 - \frac{2023\sqrt{3}}{2} + \frac{a}{2} + \frac{b\sqrt{3}}{2}) + (\frac{1999}{2} + 1005\sqrt{3} - \frac{a\sqrt{3}}{2} - \frac{b}{2})i = 0$. Thus, comparing real and imaginary parts yields $\frac{a}{2} + \frac{b\sqrt{3}}{2} = 1017 + \frac{2023\sqrt{3}}{2}$ and $\frac{a\sqrt{3}}{2} + \frac{b}{2} = \frac{1999}{2} + 1005\sqrt{3}$. Adding the two equations yields $(a+b)(\frac{1}{2} + \frac{\sqrt{3}}{2}) = \frac{4033}{2} + \frac{4033\sqrt{3}}{2}$, and thus $a+b = \boxed{4033}$. (Alternatively, we could just solve the system explicitly to obtain $a = 1998 - 12\sqrt{3}$ and $b = 2035 + 12\sqrt{3}$.)

4. (#4-NT) A positive integer N is called k-digitfriendly if N can be written as the sum of exactly k integers, not necessarily distinct, that all have the same sum of digits. For example, 31 is 2-digitfriendly because $31 = 11 + 20$, while 51 is 5-digitfriendly because $51 = 3 + 12 + 12 + 12 + 12$. Find the smallest positive integer that is both 2021-digitfriendly and 2022-digitfriendly.

Answer: 6063.

Solution: First, observe that 6063 is 2021-digitfriendly, since $6063 = 2021 \cdot 3$, and it is also 2022-digitfriendly, since $6063 = 1573 \cdot 1 + 449 \cdot 10$ is the sum of 2022 integers each with a digit sum of 1.

We claim that no smaller integer is both 2021-digitfriendly and 2022-digitfriendly. To see this, first note that any integer is congruent modulo 9, hence also modulo 3, to the sum of its digits. Therefore, if S_{2021} is the common digit sum of the terms in the 2021-digitfriendly sum for N, and S_{2022} is the common digit sum of the terms in the 2022-digitfriendly sum for N, then $N \equiv 2021S_{2021} \equiv 2022S_{2022} \pmod 3$.

In particular, since 2022 is divisible by 3, this means $N \equiv 2S_{2021} \equiv 0 \pmod 3$, so $2S_{2021}$ and therefore S_{2021} is a multiple of 3, and is therefore at least 3. Then $N = 2021S_{2021} \geq 2021 \cdot 3 = 6063$, so since 6063 works, this means $\boxed{6063}$ is the smallest possible value, as claimed.

5. (#5-AL) Suppose that a, b, c are real numbers such that $a + b + c = 9$ and $ab + ac + bc = 24$. Prove that $1 \leq \min(a, b, c) \leq 2$.

Solution 1: Imagine fixing the value of a and solving for b and c. The given equations yield $b + c = 9 - a$ and also $bc = 24 - a(b+c) = 24 - a(9-a) = 24 - 9a + a^2$. This is a quadratic system of the form $b + c = s$ and $bc = p$, which is equivalent to the equation $(x-b)(x-c) = x^2 - sx + p$, so we see it will have solutions for b, c precisely when the discriminant $s^2 - 4p \geq 0$. Plugging in $s = 9 - a$ and $p = 24 - 9a + a^2$ shows that for a fixed a, there are real solutions for b, c precisely when $(9-a)^2 - 4(24 - 9a + a^2) \geq 0$, which reduces to $-15 + 18a - 3a^2 \geq 0$. Factoring yields $-3(a-5)(a-1) \geq 0$, which is true precisely when $1 \leq a \leq 5$.

Therefore, since the variables a, b, c play symmetric roles, we must also have $1 \leq b \leq 5$ and $1 \leq c \leq 5$. This establishes the lower bound $1 \leq \min(a, b, c)$.

To obtain the upper bound, we can write down the actual values for b, c explicitly: since $b \leq c$ we see that b must get the minus sign, and so we have $b = \frac{(9-a) - \sqrt{3(a-1)(5-a)}}{2}$ and $c = \frac{(9-a) + \sqrt{3(a-1)(5-a)}}{2}$. When $a = 1$, we see $b = c = 4$, and as a increases, b and c will vary continuously in the interval $[1, 5]$. Since $a \leq b$, we see that the maximum allowed value of a will occur when $a = b$. If $a = b$ then we obtain the system $2a + c = 9$, $a^2 + 2ac = 24$, yielding $c = 9 - 2a$ and so $a^2 + 2a(9 - 2a) = 24$. This gives $3a^2 - 18a + 24 = 0$ which factors as $3(a-2)(a-4) = 0$, meaning that $a = 2$ or $a = 4$. Considering the graph of $b - a$ shows that $a < b$ when $1 \le a < 2$, and $a > b$ when $2 < a < 4$. Therefore, the greatest value of a for which $a = \min(a, b, c)$ is $a = 2$.

Thus, we see $1 \le \min(a, b, c) \le 2$, as claimed.

Solution 2: As in Solution 1, imagine fixing the value of a and think of b, c as variables in terms of a, where we impose the ordering $a \le b \le c$. Then since $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc) = 33$, we see that $-\sqrt{33} \le a \le \sqrt{33}$. If we move a from its minimum value to its maximum value, then b, c will be continuous functions of a, so $\min(a, b, c)$, being a continuous function on a closed interval, will achieve its minimum and maximum values somewhere. The minimum and maximum values must both occur at a place where two of a, b, c are equal: otherwise (say if $a < b < c$) because $a < b$ we could increase a slightly (which would increase the minimum) and because $b < c$ we could decrease a slightly (which would decrease the minimum).

We are therefore reduced to the situation where two variables are equal. If $a = b$ then we obtain the system $2a + c = 9$, $a^2 + 2ac = 24$, yielding $c = 9 - 2a$ and so $a^2 + 2a(9 - 2a) = 24$. This gives $3a^2 - 18a + 24 = 0$ which factors as $3(a - 2)(a - 4) = 0$, meaning that $a = 2$ or $a = 4$, yielding the triples $(2, 2, 5)$ and $(4, 4, 1)$. Since there are only two extremal triples, one of them must yield the smallest possible value of $\min(a, b, c)$ while the other yields the largest possible value. We conclude that $1 \le \min(a, b, c) \le 2$, as claimed.

6. (#6-TR) The expression $\cos(6°)\cos^2(12°)\tan(42°)\sin^2(66°)$ can be written as $A\sec^2(B°)$ where A is a rational number and B is an integer with $0 \le B \le 89$. Compute the ordered pair (A, B).

Answer: $(\frac{1}{128}, 84)$.

Solution: Note that $\tan(42°) = \cot(48°)$ and $\sin(66°) = \cos(24°)$, so the given expression is also equal to $\cos(6°) \cdot \cos^2(12°) \cdot \cos^2(24°) \cdot \tan(48°)$. If x is the value of this expression, then

$$\begin{aligned}
x \cdot \sin(6°) &= \sin(6°)\cos(6°) \cdot \cos^2(12°) \cdot \cos^2(24°) \cdot \cot(48°) \\
&= \frac{1}{2}\sin(12°)\cos^2(12°) \cdot \cos^2(24°) \cdot \cot(48°) \\
&= \frac{1}{4}\cos(12°) \cdot \sin(24°)\cos^2(24°) \cdot \cot(48°) \\
&= \frac{1}{8}\cos(12°) \cdot \cos(24°) \cdot \sin(48°)\cot(48°) \\
&= \frac{1}{8}\cos(12°) \cdot \cos(24°) \cdot \cos(48°).
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
x \cdot \sin(6°) \cdot \sin(12°) &= \frac{1}{8}\sin(12°)\cos(12°) \cdot \cos(24°) \cdot \cos(48°) \\
&= \frac{1}{16}\sin(24°) \cdot \cos(24°) \cdot \cos(48°) \\
&= \frac{1}{32}\sin(48°)\cos(48°) \\
&= \frac{1}{64}\sin(96°) = \frac{1}{64}\cos(6°)
\end{aligned}$$

and so $x \cdot \sin^2(6°)\sin(12°) = \frac{1}{64}\sin(6°)\cos(6°) = \frac{1}{128}\sin(12°)$. Cancelling the factor of $\sin(12°)$ yields $x \cdot \sin^2(6°) = \frac{1}{128}$, and thus $x = \frac{1}{128}\csc^2(6°) = \frac{1}{128}\sec^2(84°)$. Therefore, we can take $(A, B) = \boxed{(\frac{1}{128}, 84)}$.