Vermont State Mathematics Coalition Talent Search -- January 2022
Test 3 of the 2021-2022 school year

PRINT NAME:_____ Signature:_____
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School_____ Grade _____

Current Mathematics Teacher: _____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **February 15, 2022** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Assistant Teaching Professor at Northeastern University).

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1. Evan is coloring a map of the 14 counties in Vermont. Each county will be colored yellow, green, or blue; adjacent counties are allowed to have the same color. If a total number A of these maps have an odd number of green counties, and a total number B of these maps have an even number of green counties, determine the value of $A - B$.

Answer:_____

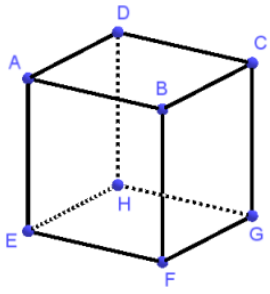
2. If a and b are integers such that $(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 641 + 632\sqrt[3]{10}$, what is the value of $a + b$?

Answer:_____

3. This is a relay problem. The answer to each part will be used in the next part.
   (a) Suppose that the polynomial $(x^2 + ax + 2021)(x^2 + bx + 2022)$ has four distinct integer roots x. What is the least possible value for $|a - b|$?
   (b) Let A be the answer to part (a). A set of integers is called *hexaphobic* if it contains no pair of distinct elements (a, b) such that $a + b$ is divisible by 6. What is the greatest possible number of elements in a hexaphobic subset of $\{1, 2, 3, ..., A\}$?
   (c) Let B be the answer to part (b). A *repunit* is a positive integer whose base-10 digits are all equal to 1, such as 111 or 1111111? How many digits does the smallest repunit divisible by $3B$ have?

Answers: (a) _____ (b) _____ (c) _____

4. Cube $ABCDEFGHI$, shown below, has side length 1. If P is the polyhedral region that lies inside both tetrahedron $ACFH$ and tetrahedron $BDEG$, what is the radius of the largest sphere that can be inscribed in P?



Answer:_____

5. In triangle ABC, $AB = 6$, $BC = 8$, and $\cos A \sin C + \sin 2C = \cos A \sin B + \sin 2B$. Find all possible values for the area of triangle ABC.

Answer:_____

6. A rectangular 9 x 2021 gameboard is colored in a black-and-white checkerboard pattern (with adjacent squares colored different colors) such that the four corner squares are black. Tinsley then places 2021 checkers on black squares of the board, one in each of the 2021 columns, such that the checker squares in adjacent columns share a vertex. Prove that the number of possible ways Tinsley can place the checkers on the board is divisible by 5^{500}.

Note: For this problem, please include your proof on separate sheets of paper.

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