Vermont Mathematics Talent Search, Solutions to Test 1, 2022-2023

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1. In the cross-number puzzle below, each entry is a digit from 1-9. Solve the puzzle:

   Across:

   1. A power of 2.
   4. A multiple of 11.
   5. A multiple of 17.

   Down:

   1. A prime.
   2. A perfect square.
   3. A power of 5.

1	2	3
4		
5		

   Answer:

[1] 2	[2] 5	[3] 6
[4] 5	7	2
[5] 7	6	5

   Solution: Since no zero digits are allowed, the only possible powers of 2 (for 1-across) are 128, 256, and 512, while the only possible powers of 5 (for 3-down) are 125 and 625. The only compatible choices are 256 for the power of 2 and 625 for the power of 5. Next, for 2-down, the perfect square must start with a 5, so it is either 529 or 576. However, 529 is not possible, because 4-across would be a multiple of 11 ending in -22, but the only such 3-digit number is 022. Therefore, 2-down must be 576. Then since the only multiple of 11 ending in -72 is 572, 4-across is 572, and likewise, the only multiple of 17 ending in -65 is 765. Then 1-down is 257, which is indeed a prime number. This yields the full grid given above.

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) A set of positive integers is called "divisible" if each element in the set is divisible by the smallest element in the set. For example, the sets $\{2, 4, 16\}$, $\{1, 3, 30\}$, and $\{12\}$ are all divisible. How many different nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are divisible?

   Answer: 541.

   Solution: We count based on the smallest element s. If $s = 1$, any of $\{2, 3, \ldots, 9\}$ can also be included for a total of 2^9 sets. If $s = 2$, any of $\{4, 6, 8, 10\}$ can also be included for a total of 2^4 sets. If $s = 3$, any of $\{6, 9\}$ can be included for a total of 2^2 sets. If $s = 4$, then 8 can or cannot be included for 2^1 sets. If $s = 5$ then 10 can or cannot be included for 2^1 sets. Finally, if $s = 6, 7, 8, 9, 10$ then the set must just be $\{s\}$. In total we get $2^9 + 2^4 + 2^2 + 2^1 + 2^1 + 5 = \boxed{541}$ divisible sets.

   (b) Let A be the answer to part (a). If a, b, and c are all positive integer divisors of $A - 1$, and $a + b = c$ where a is even and b is odd, what is the largest possible value of a?

   Answer: 120.

   Solution: Since a is even and b is odd, then c must also be odd. Then since $A = 540 = 2^2 \cdot 3^3 \cdot 5$, we see that the largest possible value of c is $3^3 \cdot 5 = 135$. Then a must be an even divisor of 540 less than 135, and the largest such divisor is 108. But this value of a works, since we can take $b = 27$. Thus the largest possible a is $a = \boxed{108}$.

   (c) Let B be the answer to part (b). A cone with vertex angle measuring $(B + 12)^\circ$ is inscribed in a sphere of radius $B/9$. What is the volume of the cone?

   Answer: 216π.

Solution: Since $B + 12 = 120$, the vertex angle is $120°$. Label the vertex as V, and let the sphere's center be O. Then OV passes through the center X of the cone's base. If A is any point on the circular base of the cone, then $m\angle OVA = 60°$ and also since $OA = OV = B/9 = 12$, in fact $\triangle OVA$ is equilateral with side length 12. Then since AXV is a 30-60-90 right triangle and $VA = 12$, we have $VX = 6$ and $AX = 6\sqrt{3}$, and thus the volume of the cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6\sqrt{3})^2 \cdot 6 = \boxed{216\pi}$.

3. Diaz has 54 fair 6-sided dice. One day, he stacks 27 of them a $3 \times 3 \times 3$ cube shape, with all of the dice in the same orientation, and then separates them again and mixes them with his other 27 standard dice. However, while stacked, the numbers on all of the interior faces of the dice have worn off, leaving them blank. (The numbers visible on the exterior of the $3 \times 3 \times 3$ cube are unaffected.) Diaz picks a random die from his collection of 54 and rolls it 3 times, obtaining no blank faces. What is the probability that his die actually has at least one blank face?

Answer: 53/1025.

Solution: Of the damaged dice, 8 have three faces blank (the vertex dice), 12 have four faces blank (the edge dice), 6 have five faces blank (the face dice), and 1 has all six faces blank (the center die).

First we find the probability of rolling no blank faces. Clearly, this cannot happen if Diaz picks the center die.

- If Diaz picks a normal die, probability 27/54, he will always roll no blank faces, for a total contribution of 1/2.
- If Diaz picks a vertex die, probability 8/54, he will roll no blank faces with probability $1/2^3$, for a total contribution of $(8/54) \cdot (1/2^3) = 1/54$.
- If Diaz picks an edge die, probability 12/54, he will roll no blank faces with probability $1/3^3$, for a total contribution of $(12/54) \cdot (1/3^3) = 2/243$.
- If Diaz picks a face die, probability 6/54, he will roll no blank faces with probability $1/6^3$, for a total contribution of $(6/54) \cdot (1/6^3) = 1/1944$.

In total the probability of rolling no blank faces is $\frac{1}{2} + \frac{1}{54} + \frac{2}{243} + \frac{1}{1944} = \frac{1025}{1944}$ while the total probability of rolling no blank faces while having a damaged die is $\frac{1}{54} + \frac{2}{243} + \frac{1}{1944} = \frac{53}{1944}$. Thus the conditional probability that the die actually has at least one blank face given that no blank faces were rolled is $\frac{53/1944}{1025/1944} = \boxed{\frac{53}{1025}}$.

4. Let F_n denote the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Quadrilateral $VMTS$ is inscribed in circle O, where $VM = MT = 2F_{2022}$, $TS = 2F_{2023}$, and $SV = 2F_{2024}$. The area of quadrilateral $VMTS$ can be written in the form $F_a F_b \sqrt{F_c}$ where a, b, and c are positive integers. Find the value of $a + b + c$.

Answer: 4051 ($a = 2022$, $b = 2025$, $c = 4$).

Solution: To compute the area of $VMTS$ we use Brahmaguptra's theorem, which says that the area of a cyclic quadrilateral (a quadrilateral inscribed in a circle) with side lengths a, b, c, and d is equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{a+b+c+d}{2}$ is the semiperimeter. Applying the theorem here with $a = b = 2F_{2022}$, $c = 2F_{2023}$, and $d = 2F_{2024} = 2F_{2022} + 2F_{2023}$ yields $s = \frac{a+b+c+d}{2} = 3F_{2022} + 2F_{2023}$, so we obtain an area of

$$\begin{aligned}
\sqrt{(s-a)(s-b)(s-c)(s-d)} &= \sqrt{(F_{2022} + 2F_{2023})(F_{2022} + 2F_{2023})(3F_{2022})(F_{2022})} \\
&= (F_{2022} + 2F_{2023})F_{2022}\sqrt{3} \\
&= (F_{2023} + F_{2024})F_{2022}\sqrt{3} \\
&= F_{2025}F_{2022}\sqrt{F_4}
\end{aligned}$$

since $F_4 = 3$. Therefore, we can take $a = 2025$, $b = 2022$, and $c = 4$, yielding $a + b + c = \boxed{4051}$.

5. Due to rampant inflation and a coin shortage, it is very difficult to make change in Kiranistan. Currently, there are only three types of bills available, and they are respectively worth 1935∃, 2021∃, and 2115∃. Evan has a large supply of each bill denomination, and must make exact payments in all his purchases due to government regulations.

   (a) Prove that Evan cannot make exact payment for a purchase valued at 175,819∃.
   (b) Prove that Evan can make exact payment for any purchase valued at least 175,820∃.

   Motivation: Note that $1935 = 43 \cdot 45$, $2021 = 43 \cdot 47$, and $2115 = 45 \cdot 47$. We can therefore obtain substantial information about the possible payment combinations by reducing calculations modulo 43, 45, and 47, since two of the three bill denominators in each case will reduce to zero.

   Solution (a): Suppose Evan uses a total of a bills worth 1935∃, b bills worth 2021∃, and c bills worth 2115∃ and that he wants to pay for a purchase valued at 175,819∃: then we must have a solution to the equation $1935a + 2021b + 2115c = 175{,}819$ where a, b, c are nonnegative integers. Reducing both sides modulo 43 yields $6c \equiv 35 \equiv -6 \pmod{43}$ which has the unique solution $c \equiv -1 \pmod{43}$. Reducing both sides modulo 45 yields $41b \equiv 4 \equiv -41 \pmod{45}$ which has the unique solution $b \equiv -1 \pmod{45}$, and reducing both sides modulo 47 yields $8a \equiv 39 \equiv -8 \pmod{47}$ which has the unique solution $a \equiv -1 \pmod{47}$.
   Therefore, since a, b, c are all nonnegative, we would have $a \geq 46$, $b \geq 44$, and $c \geq 42$. But $46 \cdot 1935∃ + 44 \cdot 2021∃ + 42 \cdot 2115∃ = 266{,}764∃$, which is far larger than the required purchase value. Therefore, Evan cannot make exact payment for the claimed purchase.

   Solution (b): To show all larger payments are possible, we use the following result, often called the postage stamp theorem or the Chicken McNuggets theorem: if a and b are relatively prime positive integers, the largest integer that cannot be written in the form $pa + qb$ for $p, q \geq 0$ is $ab - a - b$.
   Applying the postage stamp theorem with $a = 43$ and $b = 45$ shows that any integer exceeding $43 \cdot 45 - 43 - 45 = 1847$ can be written in the form $43p + 45q$. Multiplying by $c = 47$ shows that we can write any multiple of 47 exceeding $47 \cdot 1847 = 86809$ as $47(43p + 45q) = 2021p + 2115q$: in other words, we can make it using only the 2021∃ and 2115∃ bills.
   Now since $1935 = 43 \cdot 45$ is relatively prime to 47, the multiples $\{0, 1935, 2 \cdot 1935, \ldots, 46 \cdot 1935\}$ are all distinct modulo 47, so every integer modulo 47 is congruent to exactly one of them modulo 47.
   Now suppose $N > 175819$. Then since N is congruent to one of those 47 multiples $k \cdot 1935$, and since $86809 + 46 \cdot 1935 = 175819$, the difference $N - k \cdot 1935$ is a multiple of 47 exceeding 86809, so by the above it is of the form $2021p + 2115q$. Therefore $N = 1935k + 2021p + 2115q$, meaning that Evan can make exact payment for the purchase, as claimed.

6. Find the number of polynomials $p(x)$ of degree 8, all of whose coefficients are positive nonzero digits 1-9 inclusive, such that $p(x)$ is divisible as a polynomial by $x^2 - (i\sqrt{3})x - 1$ where $i^2 = -1$. (For example, one such polynomial is $p(x) = x^8 + 5x^7 + 5x^6 + 9x^5 + 7x^4 + 9x^3 + 6x^2 + 4x + 2$.)

   Answer: 7344.

   Solution: Since $p(x)$ has real coefficients we see that $p(x)$ must also be divisible by the conjugate polynomial $x^2 + (i\sqrt{3})x - 1$ and hence is divisible by their product, which is $[x^2 - (i\sqrt{3})x - 1][x^2 + (i\sqrt{3})x - 1] = x^4 + x^2 + 1$. Note in particular that this polynomial $x^4 + x^2 + 1$ has the property that $(x^2 - 1)(x^4 + x^2 + 1) = x^6 - 1$. Thus, the remainder upon dividing x^6 by $x^4 + x^2 + 1$ will be 1, the remainder of x^7 will be x, and the remainder of x^8 will be x^2. We can also easily see that the remainder upon dividing x^5 by $x^4 + x^2 + 1$ is $-x^3 - x$ and for x^4 it is $-x^2 - 1$.
   Therefore, if $p(x) = a_8x^8 + a_7x^7 + \cdots + a_1x + a_0$, dividing $p(x)$ by $x^4 + x^2 + 1$ yields a remainder of $(a_3 - a_5)x^3 + (a_2 - a_4 + a_8)x^2 + (a_1 - a_5 + a_7)x + (a_0 - a_4 + a_6)$. (One may check that the quotient in this case is $a_8x^4 + a_7x^3 + (a_6 - a_8)x^2 + (a_5 - a_7)x + (a_4 - a_6)$.) In order for the remainder to be zero we require $a_4 = a_2 + a_8 = a_0 + a_6$ and $a_3 = a_5 = a_1 + a_7$. Since these two sets of conditions are independent we may tally the number of solution tuples for each separately.
   For $a_4 = a_2 + a_8 = a_0 + a_6$, if $a_4 = n$ then there are $n - 1$ options each for (a_0, a_6) and (a_2, a_8), namely $(1, n-1)$, $(2, n-2)$, ... , $(n-1, 1)$ yielding $(n-1)^2$ choices for the pair. Summing from $n = 1$ to $n = 9$ yields $0^2 + 1^2 + \cdots + 8^2 = 204$ selections of $(a_0, a_2, a_4, a_6, a_8)$.
   For $a_3 = a_5 = a_1 + a_7$, if $a_3 = a_5 = n$ then there are $n - 1$ options for (a_1, a_7), so summing from $n = 1$ to $n = 9$ yields $0 + 1 + \cdots + 8 = 36$ selections of (a_1, a_3, a_5, a_7).
   Therefore in total we see that there are $204 \cdot 36 = \boxed{7344}$ possible $p(x)$.