Bell Inequalities and XOR Games

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Math Beyond the Horizon

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Encoding Information

Information Theory: Computation, Communcation & Cryptography

Classical Bit: 0 or 1, "On" or "Off" switch

Encode Info in strings of 0 & 1, e.g., 1101000110

Instead of "On/Off" use electron spin – little magnet

$0 \sim \uparrow$ spin up (North) $1 \sim \downarrow$ spin down (South)

BUT also $\rightarrow$ spin right (East) $\leftarrow$ spin left (West) OR $\nearrow$

What does this mean?? Can you rep any direction??

Represent with vectors in 2-dim $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

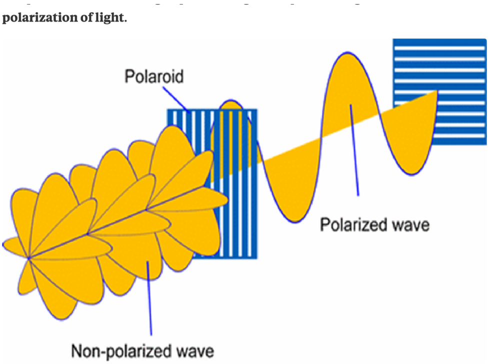
Unit of quantum info "qubit" – many different physical methods

Polarized Light

Light composed of particles called photons

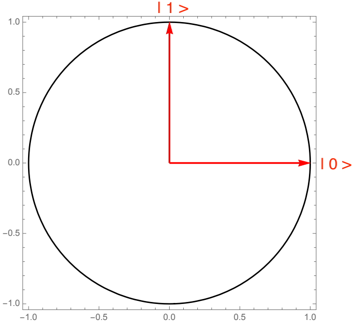
Single photon also has polarization horiz $\rightarrow \sim$ 0 vert $\uparrow \sim$ 1

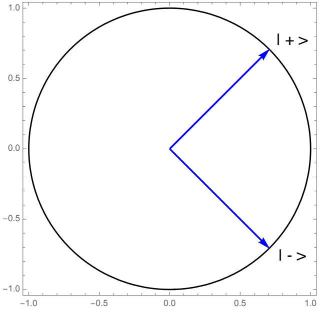
Can rotate filters in other directions, e.g., $\nearrow$ need $0 \perp 1$



Basic Qubit states

	$\vert 0\rangle$	$\vert 1\rangle$	$\vert +\rangle$	$\vert -\rangle$
$\begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
spin	$\uparrow$	$\downarrow$	$\rightarrow$	$\leftarrow$
polarization	$\rightarrow$	$\uparrow$	$\nearrow$	$\searrow$





Vector spaces

Operations $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} x+w \\ y+z \end{pmatrix} \qquad a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix}$

single qubit use 2-dim vector spaces $\quad \vec{v} = \mathbf{v} = \begin{pmatrix} x & y \end{pmatrix} = \langle v|$

Key Property of vector space $\mathcal{V}$ is

$$|u\rangle, |v\rangle \text{ in } \mathcal{V} \Longrightarrow a|u\rangle + b|v\rangle \text{ in } \mathcal{V}$$

Notation: row vecs $\langle u| = \begin{pmatrix} u_1 & u_2 \end{pmatrix}$ col vecs $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Def: inner product (dot prod) $\quad \mathbf{u} \cdot \mathbf{v} = \langle u, v\rangle \equiv u_1 v_1 + u_2 v_2$

Def: orthogonal (perpendicular) $\quad \langle u, v\rangle = 0$

Unit vectors in 2-dim, $\langle u, v\rangle =$ cos angle between $|u\rangle$ and $|v\rangle$

Arrows help visualize inner prod – only need points on unit circle

Need higher dim, e.g, two qubits use 4-dim vectors – can't draw

Principles of Quantum Mechanics

- States are represented by (unit) vectors $|v\rangle$ $\qquad x^2 + y^2 = 1$
- Time development given by Schrödinger equation (don't need)
- A physical observable Q is represented by operator (matrix) associated with special numbers q_k (called eigenvalues) and vectors (called eigenvectors) which satisfy $\quad Q|u_k\rangle = q_k|u_k\rangle$
- The result of measuring observable Q is one of the numbers q_k. System orig in state $|v\rangle$ ends in $|u_k\rangle$ with probability $|\langle v, u_k\rangle|^2$

$\Rightarrow$ Measurement destroys original state !! $\Rightarrow$ No Cloning !!

$\Rightarrow$ Overall sign doesn't matter $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} -x \\ -y \end{pmatrix}$ rep same state

Consequence of Measurement

Start with unknown $|\nearrow\rangle$ direction

Can choose measurement that decides between North and South

+1 North $|\uparrow\rangle$ -1 South $|\downarrow\rangle$

but now left in N $|\uparrow\rangle$ or S $|\downarrow\rangle$

Can't get any useful info about East or West

OR Can choose measurement that decides between East and West

but now left in E $|\rightarrow\rangle$ or W $|\leftarrow\rangle$

Can't get any useful info about North or South

Measurement process destroys orig. state – can't go back & forth

Qubits can store more info, but you can NOT extract more info

$Q|u_k\rangle = q_k|u_k\rangle$

Example: Swap $\quad X\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \qquad X\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$X\begin{pmatrix} 1 \\ 1 \end{pmatrix} = +1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad X\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Aside: evals $a_k = +1, -1$ often correspond to states $|0\rangle, |1\rangle$

$|v\rangle$ and $-|v\rangle$ rep same state

Fact: If $Q|u_k\rangle = q_k|u_k\rangle$ and $q_j \neq q_k$ then $\langle u_j, u_k\rangle = 0$

Means: Measurements only distinguish orthogonal states

$$\langle -, 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \neq 0 \;\Rightarrow \text{can't distinguish}$$

Assume: If $|v\rangle, |w\rangle$ satisfy $\langle v, w\rangle = 0$ some meas can distinguish

Two qubits need tensor products

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \qquad \langle u \otimes v, w \otimes z \rangle = \langle u, w \rangle \langle v, z \rangle$$

For matrices or operators $\quad (A \otimes B)|u \otimes v\rangle = A|u\rangle \otimes B|v\rangle$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ NOT a product state ENTANGLED

Used Key Property $\quad |u\rangle, |v\rangle \text{ in } \mathcal{V} \Longrightarrow a|u\rangle + b|v\rangle \text{ in } \mathcal{V}$

Four Entangled Bell States

$\frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$ $\qquad$ $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\qquad$ rotate invariant

$\frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$ $\qquad$ $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

Another orthog pair $|\widehat{0}\rangle = \begin{pmatrix} c \\ s \end{pmatrix}$ $\quad |\widehat{1}\rangle = \begin{pmatrix} s \\ -c \end{pmatrix}$ $\qquad c^s + s^2 = 1$, $\langle \widehat{0}, \widehat{1} \rangle = 0$

$$\frac{1}{\sqrt{2}}\big(|\widehat{0}\,\widehat{0}\rangle + |\widehat{1}\,\widehat{1}\rangle\big) = \frac{1}{\sqrt{2}}\begin{pmatrix} c \\ s \end{pmatrix} \otimes \begin{pmatrix} c \\ s \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} s \\ -c \end{pmatrix} \otimes \begin{pmatrix} s \\ -c \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} c^2 \\ cs \\ sc \\ s^2 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} s^2 \\ -sc \\ -cs \\ c^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

rotationally invariant $\qquad$ also true for $\frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$

Consequences of Entanglement

Alice and Bob share entangled state $\frac{1}{\sqrt{2}}\big(|0_A 0_B\rangle + |1_A 1_B\rangle\big)$

Suppose Alice does a 0, 1 measurement and Bob does nothing

If A gets $+1$ $|0_A\rangle$, combined system is state $|0_A 0_B\rangle$

If A gets -1 $|1_A\rangle$, combined system is state $|1_A 1_B\rangle$

Alice also knows what Bob has, instantly, even if light years apart

Alice and Bob share entangled state $\frac{1}{\sqrt{2}}\big(|0_A 1_B\rangle - |1_A 0_B\rangle\big)$

Suppose Alice does nothing and Bob does $0, 1$ measurment

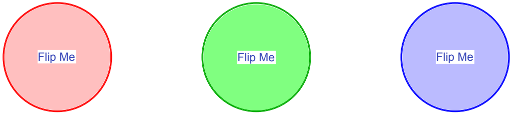
If B gets $+1$ $|0_B\rangle$, combined system is state $|1_A 0_B\rangle$

If B gets -1 $|1_B\rangle$, combined system is state $|0_A 1_B\rangle$

Bob knows what Alice has, instantly, even if light years apart

BUT does NOT $\Rightarrow$ faster than light communication

(John) Bell's Cell Phone App



- If Alice and Bob pick same color, one H - other T with prob 1
- If Alice and Bob pick diff colors, both get H with prob $\frac{3}{8}$

  When Bob gets H, Alice assumes Bob's coin is T on her side

Pick		Alice's side			Prob
A	B	R	G	B	
R	G	H	T		3/8
G	B		H	T	3/8
B	R	T		H	3/8

Bell: Three mutually exclusive event $\Rightarrow$ sum of probabilities ≤ 1.

Inequality Violation: But observed Prob is $\frac{3}{8} + \frac{3}{8} + \frac{3}{8} = \frac{9}{8} > 1$

Singlet Bell State

Alice and Bob each get half of rotationally invariant Bell state

$$|\beta\rangle = \tfrac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) = \pm\tfrac{1}{\sqrt{2}}\big(|\widehat{0}\,\widehat{1}\rangle - |\widehat{1}\,\widehat{0}\rangle\big)$$

Red, **Green**, Blue correspond to diff choices of orthog pairs

Measure with same choice, always get opposite, e.g., $|01\rangle$ or $|\widehat{1}\,\widehat{0}\rangle$

Explains first part if 0 means H 1 means T Red

Measure with another choice, $\widehat{0}$ means H $\widehat{1}$ means T **Green**

Both get H in different colors corresponds to state $|0_A\widehat{0}_B\rangle$

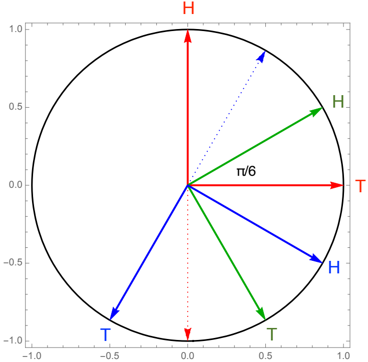
$$\text{Prob } = \big|\langle 0_A\widehat{0}_B, \beta\rangle\big|^2 = \tfrac{1}{2}\big|\langle 0_A\widehat{0}_B, 0_A 1_B\rangle - \langle 0_A\widehat{0}_B, 1_A 0_B\rangle\big|^2$$
$$= \tfrac{1}{2}\big|\langle 0_A, 0_A\rangle\langle\widehat{0}_B, 1_B\rangle - \tfrac{1}{2}\langle 0_A, 1_A\rangle\langle\widehat{0}_B, 0_B\rangle\big|^2 = \tfrac{1}{2}\big|\langle\widehat{0}_B, 1_B\rangle\big|^2$$

Prob Alice H, **Bob H** $= \tfrac{1}{2}\big|\langle H^G, T^R\rangle\big|^2$ Bob's side

Bell basis choices

Prob Alice H, **Bob H** $= \frac{1}{2}\left|\langle H^G, T^R\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\pi}{6} = \frac{1}{2}\left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{8}$

For unit vecs, $\langle u, v\rangle =$ cos of angle between



For any pair of diff colors $\langle H^C, T^D\rangle = \cos\frac{\pi}{6} \Rightarrow$ Prob of H H is $\frac{3}{8}$

XOR Games or Non-Local Games

Wizard makes rules and asks questions in form 0, 1

Team of players Alice and Bob each answer with 0, 1

Players A and B cooperate to win prize from Wizard

Players Alice and Bob communicate to decide on a strategy

Once game begins, NO communication between A and B

Called "Non-Local" because A and B can be far apart

CHSH Game

Wiz Asks			Win If		Strategy		
00	01	10	00	11	RR	RG	GR
	11		01	10		GG	

Best classical strategy, most you can win is 75% of time.

Quantum strategy can win $cos^2 \frac{\pi}{8} \approx 85\%$ How possible?

Alice and Bob share entangled state $|\beta\rangle = \frac{1}{\sqrt{2}}\big(|0_A 0_B\rangle + |1_A 1_B\rangle\big)$

Alice & Bob use different Red & Green bases $|\beta\rangle$ in A's red

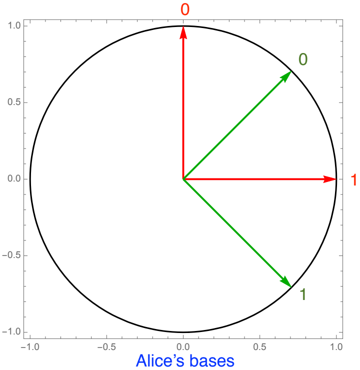
rotationally invariant $\Rightarrow$ Can assume wlog $0, 1$ in Alice's basis

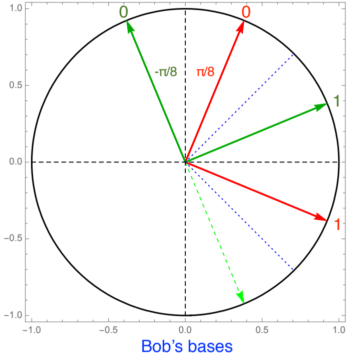
$$\text{Prob of } 0_A^R 0_B^{\widehat{G}} = \big|\langle 0_A^R 0_B^{\widehat{G}}, \beta\rangle\big|^2 = \tfrac{1}{2}\big|\langle 0_A^R 0_B^{\widehat{G}}, |0_A^R 0_B^R\rangle + \langle 0_A^R 0_B^{\widehat{G}}, |1_A^R 1_B^R\rangle\big|^2$$
$$= \tfrac{1}{2}\big|\langle 0_A^R, 0_A^R\rangle\langle 0_B^{\widehat{G}}, 0_B^R\rangle + \ 0\ \big|^2 = \tfrac{1}{2}\big|\langle 0^{\widehat{G}}, 0^R\rangle\big|^2$$

Tedious: Need to check all the combs.

CHSH Strategy

Wiz asks 0,1 Try A-R, B-G Prob 00 Win $\frac{1}{2}\left|\langle 0^{\widehat{G}}, 0^{R}\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\pi}{8}$





Ask 00, 01, 10, Win 00 or 11 Ask 11, Win 01 or 10

Strategy: Wiz Asks 0, Use Red Wiz Asks 1, Use **Green**

Win Prob: Each Comb $\frac{1}{2}\cos^2\frac{\pi}{8}$ Total $\cos^2\frac{\pi}{8} \approx 0.853$

CHSH Inequality

$0 \to +1$
$1 \to -1$

				Product	
Wiz Asks		Strategy	Win If	W	L
00 01 10		RR RG GR	00 11	+1	−1
11		GG	01 10	−1	+1

If only possible values of numbers a_0, b_0, a_1, b_1 are ± 1

$$a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 = a_0(b_0 + b_1) + a_1(b_0 - b_1) \;=\; \pm 2$$

$$\Rightarrow \quad -2 \le \mathrm{av}(a_0 b_0) + \mathrm{av}(a_0 b_1) + \mathrm{av}(a_1 b_0) - \mathrm{av}(a_1 b_1) \le +2$$

$$\begin{aligned} \mathrm{av}(a_0 b_1) &= (+1)(\text{Prob of W}) + (-1)(\text{Prob of L }) \\ &= \cos^2 \tfrac{\pi}{8} - \sin^2 \tfrac{\pi}{8} \;=\; \cos \tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}} \\ \mathrm{av}(a_1 b_1) &= (-1)\cos^2 \tfrac{\pi}{8} + (+1)\sin^2 \tfrac{\pi}{8} = -\cos \tfrac{\pi}{4} = -\tfrac{1}{\sqrt{2}} \end{aligned}$$

$$\mathrm{av}(a_0 b_0) + \mathrm{av}(a_0 b_1) + \mathrm{av}(a_1 b_0) - \mathrm{av}(a_1 b_1)$$

$$= \; 3\tfrac{1}{\sqrt{2}} - (-\tfrac{1}{\sqrt{2}}) \;=\; 4\tfrac{1}{\sqrt{2}} \;=\; 2\sqrt{2} \;>\; 2$$

GHZ game: $0 \mapsto +1 \qquad 1 \mapsto -1$

Wizard			Winning Answers				Answer ± 1
A	B	C					Win Product
0	0	0	000	011	101	110	$+1$
0	1	1	100	010	001	111	-1
1	0	1	100	010	001	111	-1
1	1	0	100	010	001	111	-1

ABC orig. have entangled state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

Probability of particular answer is $\quad |\langle GHZ, Answer\rangle|^2$

Wiz asks 0, measure in $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$ Get ± 1

Wiz asks 1, measure to get

Note: 4 ways to win 4 ways to lose

GHZ game: WIZ asks 000

Wizard			Product ±1		Winning Answers			
A	B	C	Win	Lose	Old Form			
0	0	0	$+1$	-1	000	011	101	110
0	1	1	-1	$+1$	111	100	010	001

A, B and C share $\langle GHZ| = \frac{1}{\sqrt{2}}\big(\langle 000| + \langle 111|\big)$

Prob of Answer $|\langle GHZ, Answer\rangle|^2$ Prob Win $4\big(\frac{1}{2}\big)^2 = 1$

$$\frac{1}{4}\langle 000 + 111, (|0\rangle \pm |1\rangle) \otimes (|0\rangle \pm |1\rangle) \otimes (|0\rangle \pm |1\rangle)$$

because, e.g., $\langle 000, 010\rangle = \langle 111, 011\rangle = 0$

$$= \frac{1}{4}\langle 000, 000\rangle + \frac{1}{4}(\pm 1)_A(\pm 1)_B(\pm 1)_C\,\langle 111, 111\rangle$$

$$= \begin{cases} \frac{1}{4}(1+1) = \frac{1}{2} & \text{if product } = +1 \quad \text{Always Win} \\ \frac{1}{4}(1-1) = 0 & \text{if product } = -1 \quad \text{Never Lose} \end{cases}$$

Done!

GHZ game: WIZ asks 011

Wizard			Product ±1	
A	B	C	Win	Lose
0	0	0	+1	−1
0	1	1	−1	+1

Measure gives ± 1
Answer to Wiz
still ± 1, not $\pm i$

A use $|\pm 1\rangle$, BC use $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

$$\frac{1}{4}\langle 000 + 111, (|0\rangle \pm |1\rangle) \otimes (|0\rangle \pm i|1\rangle) \otimes (|0\rangle \pm i|1\rangle)$$

$$\begin{aligned} &= \frac{1}{4}\langle 000, 000\rangle + \frac{1}{4}(\pm 1)_A(\pm i)_B(\pm i)_C\langle 111, 111\rangle \\ &= \frac{1}{4}\langle 000, 000\rangle - \frac{1}{4}(\pm 1)_A(\pm 1)_B(\pm 1)_C\langle 111, 111\rangle \quad \text{since } i^2 = -1 \\ &= \begin{cases} \frac{1}{4}(1-1) = 0 & \text{if product } = +1 \quad \text{Never Lose} \\ \frac{1}{4}(1+1) = \frac{1}{2} & \text{if product } = -1 \quad \text{Always Win} \end{cases} \end{aligned}$$

Key $i^2 = -1$ switches $+$ to $-$ in front of prod $(\pm 1)_A(\pm 1)_B(\pm 1)_C$

Aside on complex vectors

For $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad |u\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with u_j, v_j complex numbers

dual $\langle u| = \begin{pmatrix} u_1^* & u_2^* \end{pmatrix}$ has complex conj $(a + ib)^* = a - ib$

inner product $\langle u, v \rangle = u_1^* v_1 + u_2^* v_2 \qquad\qquad i^2 = -1$

$$\langle v, v \rangle = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^* v_1 + v_2^* v_2 = |v_1|^2 + |v_2|^2$$

$$|u_+\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |u_-\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad Q|u_\pm\rangle = \pm 1 |u_\pm\rangle$$

orthogonal $\langle u_+, u_- \rangle = \tfrac{1}{2} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \tfrac{1}{2}(1 + (-i)^2) = 0$

Connes Embedding Problem

Major mathematical problem unsolved for over 40 years

Technical: Approximate operator algebras by matrix algebras??

Reformulated in many other areas of math and CS

2011 Connected to Bell Inequalities and quantum correlations

2016 Breakthrough using very complicated non-local XOR game

solved a closely related problem

2020 (Jan) Comp. science – entanglement value of XOR game

equivalent to Turing halting problem – undecidable

$\Longrightarrow$ 40 years of math problems toppled like dominoes

Connes Embedding Conjecture False

Further Reading

- N .D. Mermin *Boojums All the Way Through* (Cambridge, 1990)

delightful collection of essays, including some on Q.M.

- D. Wick *The Infamous Boundary* (Birkhaüser, 1995)

history of Bell ineq. with Appendix on Probability by W. Faris

- B. Schumacher and M. Westmoreland *Quantum Processes, Systems, and Information* (Cambridge, 2010)

excellent introductory text written for undergraduates

- N .D. Mermin *Quantum Information Science* (Cambridge, 2007)