Vermont Talent Search **March 9, 2011**
School Year 2010-2011
Test 3 Solutions

Problem 1.
Find the sum of all angles θ, $0 \le \theta \le 2\pi$, such that
$\left(8\cos 4\theta - 3\right)\left(\cot\theta + \tan\theta - 2\right)\left(\cot\theta + \tan\theta + 2\right) = 12$
Hint: Express sum is radians.

Solution:
First we have from the last two terms from the LHS

$$\left(\cot\theta + \tan\theta - 2\right)\left(\cot\theta + \tan\theta + 2\right) = \left(\cot\theta + \tan\theta\right)^2 - 4 = \cot^2\theta + \tan^2\theta - 2 = \frac{\left(1 - \tan^2\theta\right)^2}{\tan^2\theta}$$

$$\frac{\left(1 - \tan^2\theta\right)^2}{\tan^2\theta} = 4\left(\frac{1 - \tan^2\theta}{2\tan\theta}\right)^2 = \frac{4}{\tan^2 2\theta} \quad \text{using identity } \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Now inserting this result into the given equation and simplifying yields
$8\cos 4\theta - 3 = 3\tan^2 2\theta$ or
$8\cos 4\theta = 3\left(1 + \tan^2 2\theta\right) = 3\sec^2 2\theta$

Hence $\cos 4\theta \cdot \cos^2 2\theta = \frac{3}{8}$. Now since $\cos 4\theta = 2\cos^2 2\theta - 1$ we have
$2\cos^4 2\theta - \cos^2 2\theta - 3/8 = 0$. Factoring yields the following

$$2\left(\cos^2 2\theta - \frac{3}{4}\right)\left(\cos^2 2\theta + \frac{1}{4}\right) = 0$$

Thus $\cos 2\theta = \pm\frac{\sqrt{3}}{2}$.

For $+\frac{\sqrt{3}}{2}$; $2\theta = \frac{\pi}{6}$ and $\frac{11\pi}{6}$ thus $\theta = \frac{\pi}{12}, \frac{23\pi}{12}$ and $\frac{11\pi}{12}, \frac{13\pi}{12}$

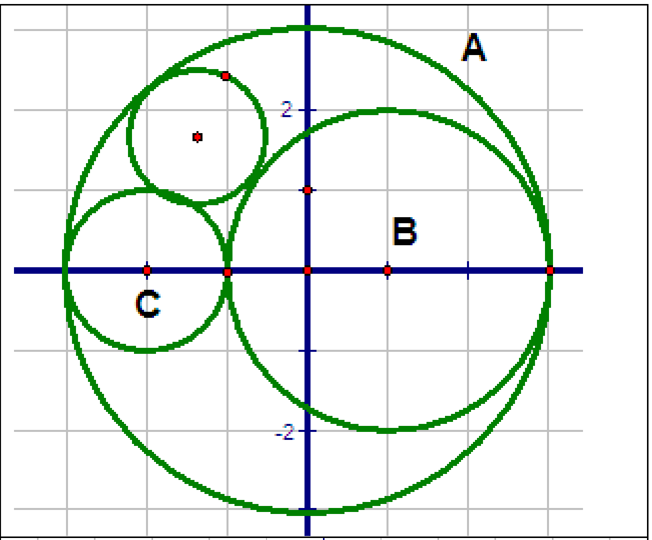
For $-\frac{\sqrt{3}}{2}$; $2\theta = \frac{5\pi}{6}$ and $\frac{7\pi}{6}$ thus $\theta = \frac{5\pi}{12}, \frac{19\pi}{12}$ and $\frac{7\pi}{12}, \frac{17\pi}{12}$

Finally, the sum of these 8 angles is $\frac{96\pi}{12} = 8\pi$

The Vermont Math Coalition is grateful to problem contributors for this test including Tony Truno, retired Burlington High School Math teacher and Evan Dummit, a graduate mathematics student at the University of Wisconsin, Madison WI.

Problem 2.
Four circles are drawn such that each circle is tangent to the other three circles. The radii of the three larger circles are 1, 2 and 3. Find the smallest possible radius of the fourth circle.
Solution:

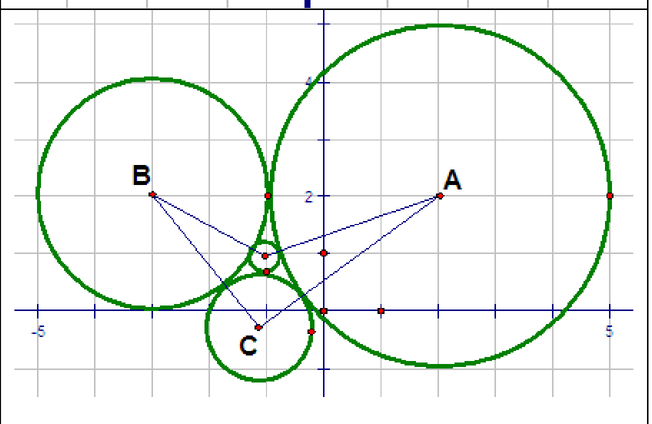
The diagram for one possible solution is shown at the right where “A” is on the circumference of the largest $(r = 3)$ circle.



But the diagram for the smallest possible circle is shown at the right.

Let the centers of the four circles be A, B, C and the center of the smallest circle be O; let the radius of the small circle be *r*. Thus $AO = r+3, BO = r+2, CO = r+1$

Note that ΔABC is a 3-4-5 right triangle.



Let angle $ACO = \theta$, then angle $BCO = 90 - \theta$, then Law of Cosines in ΔACO yields:
$(3+r)^2 = 16 + (r+1)^2 - 8(1+r)\cos\theta$ and hence $\cos\theta = \dfrac{2-r}{2(r+1)}$

In ΔBCO: $(r+2)^2 = 9 + (r+1)^2 - 6(r+1)\cos(90-\theta)$ and hence $\sin\theta = \dfrac{3-r}{3(r+1)}$

Since $\sin^2\theta + \cos^2\theta = 1$ we get $\dfrac{(2-r)^2}{2(r+1)^2} + \dfrac{(3-r)^2}{9(r+1)^2} = 1$

Simplifying: $23r^2 + 132r - 36 = 0$ or $(23r-6)(r+6) = 0$ and thus $r = \dfrac{6}{23}$.

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Problem 3.

In a narrow alley near UVM, a ladder leans against a wall at an angle of 75° with the horizontal ground and reaches a point m feet above the ground. Keeping the foot of the ladder at the same point, the top of the ladder is moved to the wall on the other side of the alley. The ladder now makes an angle of 45° with the horizontal and reaches a point n feet above ground $(m > n)$.

Find, *in simplest form,* the width of the alley in terms of m and n.

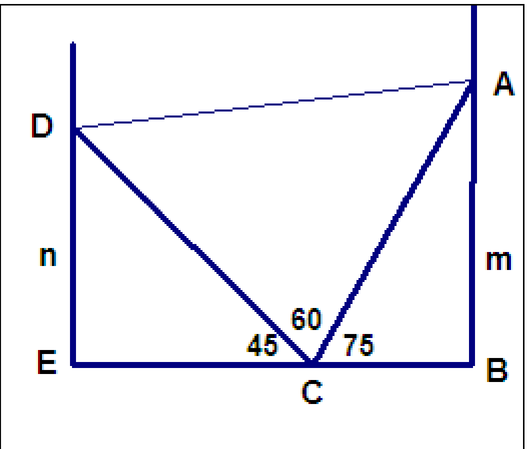
Solution:

Note ΔDCE is an isosceles right triangle with legs n.
Thus $DC = n\sqrt{2}$

On the hypotenuse of right ΔABC construct an equilateral ΔACD.

In ΔABC: $\dfrac{m}{AC} = \cos(45-30) = \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{2}\right)$

Thus $AC = \dfrac{2m\sqrt{2}}{\sqrt{3}+1}$



Since ΔACD is equilateral: $n\sqrt{2} = \dfrac{2m\sqrt{2}}{\sqrt{3}+1}$ and $n = \dfrac{2m}{\sqrt{3}+1}$

In ΔABC: $\dfrac{BC}{m} = \tan(45-30) = \dfrac{1-\dfrac{1}{\sqrt{3}}}{1+\dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ and $BC = \dfrac{m\left(\sqrt{3}-1\right)}{\sqrt{3}+1}$

So $BE = n + BC = \dfrac{2m}{\sqrt{3}+1} + \dfrac{m\left(\sqrt{3}-1\right)}{\sqrt{3}+1} = m$

Problem 4.

For $x > 0$, and $(4x)^{\log 2} - (9x)^{\log 3} = 0$, find x.

Solution:

Equate the terms on the LHS of the given equation and take logarithms; any base OK

$(\log 2)\log 4x = (\log 3)\log 9x$

$(\log 2)(\log 4 + \log x) = (\log 3)(\log 9 + \log x)$

$2(\log 2)^2 + (\log 2)(\log x) = 2(\log 3)^2 + (\log 3)(\log x)$

$\log x(\log 2 - \log 3) = 2(\log 3 + \log 2)(\log 3 - \log 2)$

$\log x = -2(\log 3 + \log 2) = \log 6^{-2}$

Thus $x = 6^{-2} = \dfrac{1}{36}$

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Problem 5.

Evaluate: $2\sum_{k=1}^{k=1005}\left[(2k-1)(2k+1)\right]^{-1}$

Solution: Note that $\frac{2}{(2k-1)(2k+1)}=\frac{1}{2k-1}-\frac{1}{2k+1}$

Thus the series is $\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+...+\left(\frac{1}{2009}+\frac{1}{2011}\right)=$

$=1-\frac{1}{2011}=\frac{2010}{2011}$

Problem 6.

Three urns collectively contain 15 red marbles and 15 blue marbles. One marble is drawn randomly from each urn. If the probability that all three marbles are blue is $\frac{11}{125}$, what is the probability that all three marbles are red?

Solution:

Let $a=$ number of blue marbles in Urn 1
$b=$ number of blue marbles in Urn 2
$c=$ number of blue marbles in Urn 3
$d=$ total number of marbles in Urn 1
$e=$ total number of marbles in Urn 2
$f=$ total number of marbles in Urn 3

Therefore we are seeking integer solutions to the following equations:

$\left(\frac{a}{d}\right)\left(\frac{b}{e}\right)\left(\frac{c}{d}\right)=\frac{11}{125}$ $,a+b+c=15$ $,d+e+f=30$

Since the denominator of the product is 125, that means each fraction's denominator must be divisible by 5, Note that 25 cannot be in any denominator because this would require more than 30 marbles; 25 in one, 5 in another which leaves none for the third.

Additionally, since there is a factor of 11 in the numerator, the corresponding denominator must be at least 15. So the possibilities are:

$\left(\frac{a}{5}\right)\left(\frac{b}{10}\right)\left(\frac{11}{15}\right)$ or $\left(\frac{a}{5}\right)\left(\frac{b}{5}\right)\left(\frac{11}{20}\right)$ In the first case our conditions require $a+b=4$ as well as $a\cdot b=6$ to which there is no integer (or real) solution. In the second case our conditions require $a+b=4$ and $ab=4$ to which there is solution $(a,b)=(2,2)$.Therefore the desired solution is $\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{11}{20}\right)=\frac{11}{125}$ and the answer is $\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{9}{20}\right)=\frac{81}{500}$

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Problem 7

Find the sum of n terms of an arithmetic progression whose first term is the sum of the first n positive integers and whose common difference is n.

Solution:

The sum of the first n positive integers is $S = \frac{(n+1)n}{2}$.

Let T be the sum of the arithmetic progression.

Then $T = S + (S+n) + (S+2n) + \ldots + (S + (n-1)n)$

$$= \left[S + (S + (n-1)n)\right]\frac{n}{2}$$
$$= \left[\frac{2S + (n-1)n}{2}\right]n$$
$$= \left[\frac{(n+1)n + (n-1)n}{2}\right]n$$
$$= n^3$$

Problem 8.

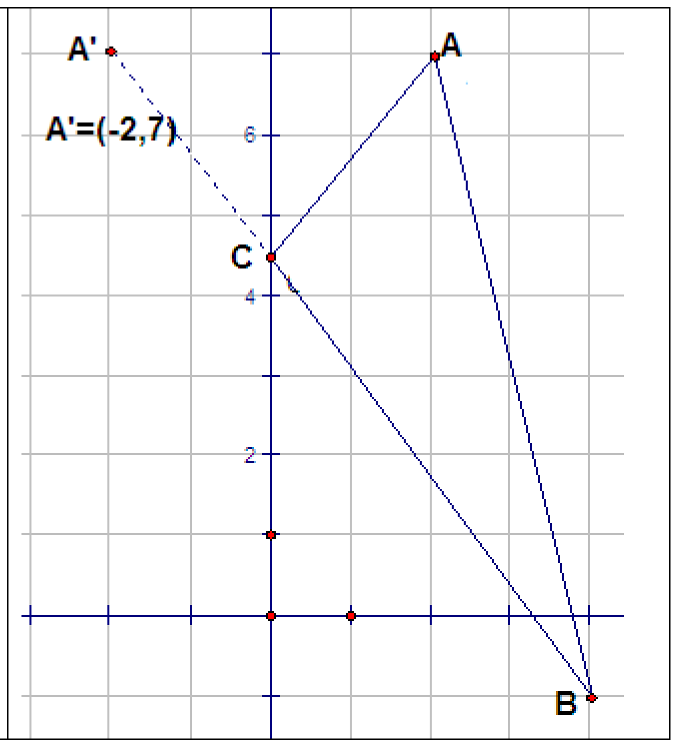
Let the three vertices of a triangle T be $(2,7), (4,-1)$ and $(0, y)$. Find y so that the perimeter of the triangle T is as small as possible.

Solution:

Let point $C = (0, y)$. Draw point A' as the mirror image of A. Thus $AC = A'C$ and $AC + CB$ is minimized when C is on the straight line $A'B$.

Slope of $A'B = \frac{-1-7}{4+2} = -\frac{4}{3}$

Thus line $A'B$ is $y + 1 = -\frac{4}{3}(x-4)$ or $y = \frac{-4x+13}{3}$ and $y(0) = \frac{13}{3}$



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