Problem 1.

The unit squares in a 3 x 3 grid are colored blue and gray at random, and each color is equally likely. What is the probability that a 2 x 2 square will be blue?

Answer: $\frac{95}{512}$

There are 4 possible 2 x 2 squares. For a single square the probability that it is blue is $\frac{1}{2^4}$. For two squares, the probability that both are blue is $\frac{1}{2^6}$ when they are adjacent squares and $\frac{1}{2^7}$ in the two cases when they are diagonally opposite. For three squares the probability that all are blue is $\frac{1}{2^8}$. The probability that all four squres are blue is $\frac{1}{2^9}$. Therefore the probability that a 2 x 2 square is blue is;

$$4\left(\frac{1}{2^4}\right)-\left(4\left(\frac{1}{2^6}\right)+2\left(\frac{1}{2^7}\right)\right)+4\left(\frac{1}{2^8}\right)-\left(\frac{1}{2^9}\right)=\frac{95}{212}$$

Problem 2.

Find the sum of the absolute values of the six distinct values for α such that $(\alpha^3-9\alpha^2)^2=8$.

Note: This problem had a typo, the original problem was $(\boldsymbol{\alpha^3-9\alpha})^{\mathbf{2}}=\mathbf{8}$ ***which was much easier to solve via simple factoring!***

Answer: $\cong 20.24$

$(\alpha^3-9\alpha^2)^2=8$ so $\alpha^3-9\alpha^2=\sqrt{8}$ *or* $\alpha^3-9\alpha^2=-\sqrt{8}$ resulting in the two equations

$$\alpha^3-9\alpha^2-\sqrt{8}=0$$

$$\alpha^3-9\alpha^2+\sqrt{8}=0$$

Using either Cardano's method or a computer algebra system we find the absolute value of the roots add up to approximately 19.16. Cardano's method is comparable to completing the square for quadratics but we do it for a cubic equation. We first reduce each equation to the form of equation (2): $t^3+pt+q=0$. At first this doesn't appear to be an easier form to solve but with additional substitution you can make equation (2) into a quadratic form and use the discriminant to determine the solutions. Using the substitution $x=t-\frac{a}{3}$ we can determine an a value which makes the x squared term go to zero, the value for our two equations is -9. Therefore we substitute $x=t+3$ into both equations and end up with the two equations;

(3): $$t^3-27t-2\sqrt{2}-54=0 \; and$$

(4): $$t^3-27t+2\sqrt{2}-54=0$$

Please note that in these equations $p=-27 \; and \; q=-2\sqrt{2}-54 \; or \; q=2\sqrt{2}-54$.

If we recognize that a perfect cube can be represented by $u^3 - v^3 = (u - v)^3 + 3uv(u - v)$ where $u^3 - v^3 = -q \; and \; 3uv = p$. This simplifies to a quadratic form of $u^6 + qu^3 - \left(\frac{p}{3}\right)^3 = 0$, the discriminant for this quadratic would be $D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$. Our solution for $u \; and \; v$ is: $u = \sqrt[3]{\left(-\frac{q}{2} + \sqrt{D}\right)}$ and $v = \sqrt[3]{\left(\frac{q}{2} + \sqrt{D}\right)}$ therefore our three x solutions in the case where $D > 0$ are

$$x_1 = t + 3 = u - v + 3$$

$$x_{2,3} = -.5(u - v) \pm (u - v)\frac{\sqrt{3}}{2}i + 3$$

For equation 3 which has a positive discriminant, our resultant roots are $x_1 = 9.0347, x_2 = -.0173 - .5593i, and \; x_3 = -.01733 + .5593i$.

For equation 4, where $D < 0$, all the roots are real and different. This case requires using DeMoivre's law to simplify the solutions of the resultant sixth degree eq1uation. Our equations simplify to;

$$x_1 = 2r^{\left(\frac{1}{3}\right)} cos\left(\frac{\phi}{3}\right) + 3$$

$$x_2 = 2r^{\left(\frac{1}{3}\right)} cos\left(\frac{\phi + 2\pi}{3}\right) + 3$$

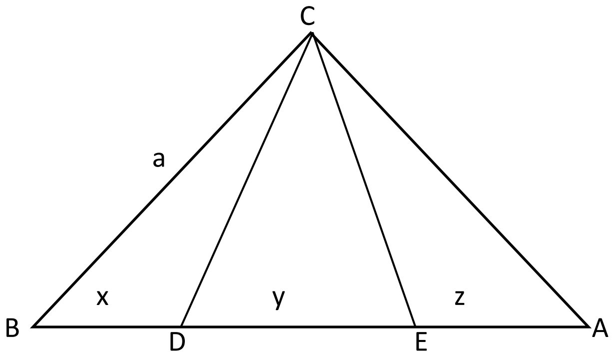
$$x_3 = 2r^{\left(\frac{1}{3}\right)} cos\left(\frac{\phi + 4\pi}{3}\right) + 3$$

$$where \; r = \sqrt{\left(\left(-\frac{p}{3}\right)^3\right)} \; and \; cos\phi = -\frac{q}{2r}$$

This results in the solutions for equation 4 of; $x_1 = 8.965, x_2 = -.5444, and \; x_3 = .5796$. Adding the absolute values of our six roots we get approximately 20.24.

Problem 3.

In triangle ABC, the trisectors of angle C divide AB into segments of lengths 4, 6, and 8 (in some order). Find all possible values of AC + BC.



Answer: $12 + 3\sqrt{10}$.

By the angle bisector theorem we have $\frac{BC}{BD} = \frac{CE}{DE}$ and $\frac{CD}{DE} = \frac{AC}{AE}$, so $CD = \frac{y}{z}b$ and $CE = \frac{y}{x}a$.

By Stewart's Theorem in triangle BCE we have $BC^2 \cdot DE + CE^2 \cdot BD = CD^2 \cdot BE + BD \cdot DE \cdot BE$ substituting in $a^2 \cdot y + \left(\frac{y}{x}a\right)^2 \cdot x = \left(\frac{y}{z}b\right)^2 \cdot (x+y) + x \cdot y \cdot (x+y)$. Factoring and than cancelling gives the following;

$a^2\left(\frac{y}{x}\right) \cdot (x+y) = \left[\left(\frac{y}{z}b\right)^2 + xy\right] \cdot (x+y)$ now cancelling yields $\frac{y}{x}a^2 = \left(\frac{y}{z}\right)^2 b^2 + xy$ so $a^2 = \frac{xy}{z^2}b^2 + x^2$.

Similarly, in triangle ACD (after interchanging a and b), $b^2 = \frac{yz}{x^2}a^2 + z^2$. Plugging in this equation to the equation from BCE yields $a^2 = \frac{xy}{z^2}\left[\frac{yz}{x^2}a^2 + z^2\right] + x^2 = \frac{y^2}{xz}a^2 + x(y+x)$ so that $a^2 = \frac{x^2z(y+x)}{xz-y^2}$. We then also obtain $b^2 = \frac{xz^2(y+z)}{xz-y^2}$.

We have the following cases;

- x=4, y = 6 and z=8 No real solutions.
- x=4, y=8 and z=6 No real solutions.
- x=6,y=4,z=8 $a = 3\sqrt{10}$ and $b = 12$
- x=6, y=8, z=4 No real solutions.
- x=8,y=4,z=6 $a = 12$ and $b = 3\sqrt{10}$
-
- x=8,y=6,z=4 No real solutions.

Therefore the solution for real values AB + BC is $12 + 3\sqrt{10}$.

Problem 4.

Find the number of palindromes which have 16 digits, in which the product of the non-zero digits is 16, and the sum of the digits is also equal to 16. How many such numbers are there?

Answer: 490

A 16 digit palindrome must have a form of $a_1a_2 \ldots a_8a_8 \ldots a_2a_1, with\ a_1 \in \{0,1,2,\ldots,9\}$ for

$1 \le i \le 8, and\ a_1 \neq 0$. For the sum of its digits to be 16, the sum $a_1 + a_2 + \cdots + a_8$ must be 8. For the product of the non-zero digits of the palindrome to equal 16, the product of the non zero digits of the first 8 digits must be 4. The first 8 digits must therefore be arrangements either of digits 2,2,1,1,1,1,0,0 or of digits 4,1,1,1,0,0,0 with 0 excluded as a first digit. For example; 4111100000011114 and 2011121001211102 are two such palindromes. If the first 8 digits contain two 2's, four 1's and two 0's, then the number of arrangements of these 8 digits is $\frac{6(7!)}{2!2!4!} = 315$. If the first digits of the palindrome contain one 4, four 1's and three 0's then the number of arrangements is $\frac{5(7!)}{4!3!} = 175$. Therefore there are 315 + 175 = 490 of such numbers.

Problem 5.

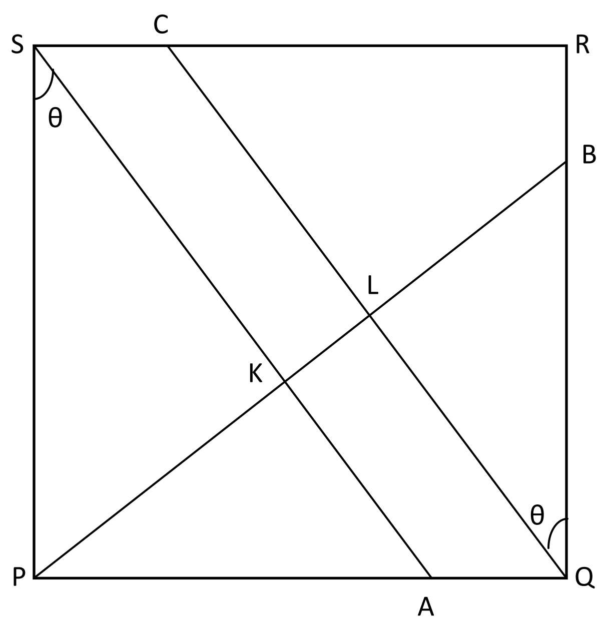
If $\log_9 A = \log_{18} B = \log_{16} C = 2013^{2014}$, find $\log_C\left(\frac{B}{A}\right)$. Answer: $\frac{1}{4}$

Suppose $\log_9 A = \log_{18} B = \log_{16} C = k$. Then $A = 9^k$, $B = 18^k$, and $C = 16^k$, so $\frac{B}{A} = 2^k$, hence by the change of base formula, $\log_C\left(\frac{B}{A}\right) = \frac{\log_2\left(\frac{B}{A}\right)}{\log_2 C} = \frac{k}{4k} = \frac{1}{4}$.

Problem 6.

Let $PQRS$ be a unit square. Define A, B, C and D to be points on PQ, QR, RS and SP respectively, such that $PA = QB = RC = SD = \frac{1999}{2000}$. Construct the triangles, QDR, RAS, SBP and PCQ. Find the area of the region which is common to all four triangles.

Answer: $A = \frac{1}{7996001}$



SA ∥ CQ and $PB \perp SA$. Let $\theta = \angle PSA$. Therefore, $\angle RQC = \theta$ so $SA = CQ = PB = \frac{1}{cos\theta}$, and $PA = QB = tan\theta$. $PK = sin\theta$, which implies that $LB = \frac{\sin^2\theta}{\cos\theta}$. $KL = PB - PK - LB$, therefore

$$KL = \frac{1}{cos\theta} - sin\theta - \frac{\sin^2\theta}{\cos\theta} = \frac{1-\sin^2\theta}{\cos\theta} - \sin\theta$$

Therefore $KL = cos\theta - sin\theta$ and the square with side length KL has area $(cos\theta - sin\theta)^2$

This reduces to $1 - 2sin2\theta = 1 - \frac{2tan\theta}{1+tan^2\theta}$ and

since $tan\theta = \frac{1999}{2000}$, the Area is $1 - \frac{2\left(\frac{1999}{2000}\right)}{1+\left(\frac{1999}{2000}\right)^2} = \frac{1}{7996001}$.

Problem 7.

If $f(x)$ is a function, its "first twist" is defined to be $f_1(x) = \frac{f(x+f(x))}{f(x)}$, and then its "$n$th twist" $f_n(x)$ is defined recursively to be the first twist of its $(n-1)st$ twist $f_{n-1}(x)$. For example, the first twist of $f(x) = 3x$ is $f_1(x) = \frac{3(x+3x)}{3x} = 4$ and the second twist is $f_2 = \frac{4}{4} = 1$. Let $p(x) = x^2 - 20x + 13$, and let $p_n(x)$ be the nth twist of $p(x)$. Find the smallest value of n such that all zeroes of $p_n(x)$ are negative.

Answer: n=20

Let $g(x) = (x-r)(x-s)$. The first twist of $g(x)$ is given by

$$\frac{g(x+(x-r)(x-s))}{(x-r)(x-s)} = \frac{[x+(x-r)(x-s)-r][x+(x-r)(x-s)-s]}{(x-r)(x-s)}$$

$$= \frac{[(x-r)(x+1-s)][(x-s)(x+1-r]}{(x-r)(x-s)}$$

$$= (x+1-s)(x+1-r)$$

Hence we see that the first twist of a monic quadratic polynomial is once again a monic quadratic polynomial whose two roots are each 1 less than the roots of the original polynomial. The roots of $p(x)$, from the quadratic formula, are $\frac{20\pm\sqrt{400-4(13)}}{2} = 10 \pm \sqrt{87}$. Since $19 < 10 + \sqrt{87} < 20$, we see that 20 iterations of the "twist" are required to make both roots negative.

Note: We have implicitly disregarded the removable discontinuities of the twist function that are introduced when dividing by $f(x)$. For the sake of completeness, we can observe (by an easy induction) that any of the twist functions will only be undefined at the zeroes of the previous twist function, and at all other values will behave as the polynomial function above, so in particular, the removable discontinuities do not cause any issues because $\sqrt{87}$ is irrational.

Problem 8.

A finite collection of positive integers has arithmetic mean 10. Find all possible integral values for the geometric mean of the integers, if

a) there are no duplicates allowed in the collection. Answer: 6, 8, 10

b) duplicates are allowed in the collection. Answer: 2, 3, 4, 5, 6, 7, 8, 9, 10

The arithmetic-geometric mean inequality says that the arithmetic mean is at least as large as the geometric mean, so we must at most check whether the integers 1, ... ,10 are possible.

a) We can also observe that for any particular geometric mean, there are only finitely many collections of distinct positive integers with that geometric mean: if $a_1 \dots . a_n = k^n$ and $a_1 < \cdots < a_n$, then $a_t \geq k$ for $t \geq k$, so $[(k-1)!] \cdot [k^{n-k}] \cdot a_n \leq [a_1 \dots a_{k-1}] \cdot [a_k \dots a_{n-1}] \cdot a_n = k^n$, so $a_n \leq \frac{k^k}{(k-1)!}$. Thus the largest integer is bounded above, so there are only finitely many tuples, which we can straightforwardly list. Now we work in cases;

- 1 is not possible, since all terms would have to be 1.
- p is not possible for any prime p, since the only possible sets with this geometric mean are $\{p\}, \{1, p^2\}$, and $\{1, p, p^2\}$, and none of these has arithmetic mean 10.
- For a prime p^2, sets with a geometric mean p must contain only powers of p, and the average exponent must be 2. This yields a short list of possibilities: $\{1, p^4\}, \{1, p^2, p^4\}, \{1, p, p^5\}, \{1, p, p^2, p^5\}, \{1, p, p^3. p^4\}, \{1, p, p^2, p^3, p^4\}, \{p, p^3\}, \{p, p^2, p^3\}$. Running through the possibilities listed above show that 4 and 9 are not possible.
- 6 is possible: $\{2,18\}$. 8 is possible: $\{4,16\}$. And 10 is possible: $\{10\}$
- This exhausts all possibilities, so the answer is 6, 8, 10.

b) As noted above, 1 is not possible. However, all of the remaining integers, 2 through 10, are in fact possible. One can construct lists by hand for this task, but here is a general construction for a sequence of integers whose geometric mean is a and whose arithmetic mean is n: use $(k-1)(n-a)$ copies of 1, $a^k + k - 1 - nk$ copies of a, and $(n-a)$ copies of a^k, where k is such that $a^k > 1 + (n-1)k$. (Such a k always exists since the exponential grows much faster than the linear function, provided $a > 1$.) This construction yields the following results for the integers not included in part (a):

- 2 ($k = 6$): 40 copies of 1, 9 copies of 2, 8 copies of 64.
- 3($k = 4$): 21 copies of 1, 44 copies of 3, 7 copies of 81.
- 4($k = 3$): 12 copies of 1, 36 copies of 4, 6 copies of 64.
- 5($k = 2$): 5 copies of 1, 6 copies of 5, 5 copies of 25.
- 7($k = 2$): 3 copies of 1, 30 copies of 7, 3 copies of 49.
- 8($k = 2$): 2 copies of 1, 45 copies of 8, 2 copies of 64.
- 9($k = 2$): 1 copy of 1, 62 copies of 9, 1 copy of 81.

Thus the answer is 2,3,4,5,6,7,8,9,10.