Problem 1.

In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

Answer: $\sqrt{51}$.

Solution: Let $m\angle A = x$ and $m\angle B = y$. Note that we have two pairs of isosceles triangles, so $m\angle A = m\angle ACD$ and $m\angle B = m\angle BCD$. Since $m\angle ACD + m\angle BCD = m\angle ACB$, we have $180^o = m\angle A + m\angle B + m\angle ACB = 2x + 2y$, hence $m\angle ACB = x + y = 90^o$. Since $\angle ACB$ is right, we can use the Pythagorean Theorem to compute BC as $\sqrt{10^2 - 7^2} = \sqrt{51}$. Note: for a shortcut, note that D is the circumcenter of ABC and lies on the triangle itself, so it must lie opposite a right angle.

Problem 2.

The increasing sequence $S = \{2, 3, 5, 6, 7, 10, 11, \ldots\}$ consists of all positive integers which are neither a perfect square nor a perfect cube. What is the 2014th term of S?

Answer: 2068

Solution. Clearly the required term n is unique and greater than 2014. Since the squares and cubes thin out as they increase, n is not likely to be very much greater than 2014. In fact the number of squares less than 2014 is 44 ($44^2 = 1936$) and the number of cubes is only 12 ($12^3 = 2197$). Thus, up to 2014 not more than 56 positive integers have failed to qualify for membership in S. Actually, since the sixth powers 1 and 64 are both squares and cubes, only 54 positive integers < 2014 are not in S, making 2014 itself the 1960th term in S. Advancing 54 terms from 2014 would take us to 2068. The 2014th term of S is 2068.

Problem 3.

Let $N = 6 + 66 + 666 + \cdots + \underbrace{666\ldots6}_{2014\ 6's}$ where the last number has 2014 sixes. What is the sum of the digits of 27N?

Answer: 18117

Solution. Call the required sum S_{2014}, so that for $n \geq 2$

$$\begin{aligned}
S_{2014} &= 6 + 66 + 666 + \cdots + \underbrace{666\ldots6}_{2014\ 6's} \\
&= (0+6) + (60+6) + (660+6) + \cdots + (666\cdots60 + 6) \\
&= 10\left(6 + 66 + \cdots + \underbrace{666\cdots6}_{2014-1\ 6's}\right) + 6(2014) \\
&= 10S_{2014-1} + 6n \\
&= 10\left(S_{2014} - \underbrace{666\cdots6}_{2014\ 6's}\right) + 6(2014)
\end{aligned}$$

Solving for S_{2014}, we obtain $9S_{2014} = \underbrace{666\cdots6}_{2014\ 6's}0 - 6(2014) = \frac{2}{3}(999\cdots90 - 9(2014))$,

$$S_{2014} = \frac{2}{3}\left(\underbrace{111\cdots1}_{2014\text{ 1's}}0 - 2014\right)$$

Since $\underbrace{111\cdots1}_{2014\text{ 1's}}0 = 10 + 10^2 + \cdots + 10^{2014} = \frac{10(10^{2014}-1)}{9}$, we may write

$$S_{2014} = \frac{2}{3}\left(\frac{10^{2014}(10^{2014}-1)}{9} - 2014\right) = \frac{2}{27}(10^{2014+1} - 10 - 9(2014))$$

Multiplying by 27 we get $S_{2014} = 2(10^{2015} - 10 - 9(2014)) = 2(10^{2015} - 18136))$. This leads to

$\underbrace{200\cdots0}_{2015\text{ 0's}} - 36272 = 1\underbrace{999\cdots9}_{2010\text{ 9's}}63728$. The sum of the digits is $1 + 2010 \cdot 9 + 6 + 3 + 7 + 2 + 8 = 18117$.

Problem 4.

Find all real x for which $1 + \sqrt{x+1} + \sqrt{2x+1} = \sqrt{7x+1}$.

Answer: 24

Solution: Rearranging yields $1 + \sqrt{x+1} = \sqrt{7x+1} - \sqrt{2x+1}$. Squaring both sides yields;

$$x + 2 + 2\sqrt{x+1} = 9x + 2 - 2\sqrt{(7x+1)(2x+1)}$$

$$\sqrt{x+1} + \sqrt{(7x+1)(2x+1)} = 4x$$

Squaring both sides again yields;

$$(x+1) + (7x+1)(2x+1) + 2\sqrt{(x+1)(7x+1)(2x+1)} = 16x^2$$

$$\sqrt{(x+1)(7x+1)(2x+1)} = x^2 - 5x - 1$$

Squaring both sides yet again yields;

$$(x+1)(7x+1)(2x+1) = (x^2 - 5x - 1)^2$$

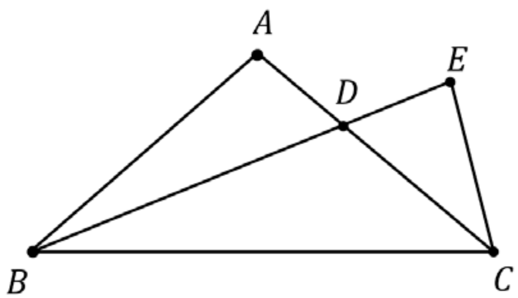
Rearranging gives:

$$x^4 - 24x^3 = 0$$

So $x = 0$ (which is extraneous), or $x = 24$.

Problem 5.

In isosceles triangle *ABC* the base angles at *B* and *C* are $40°$. The bisector of angle *B* meets *AC* at *D* and *BD* is extended to *E* so that $DE = AD$ (see figure below). How big is $\angle E$?

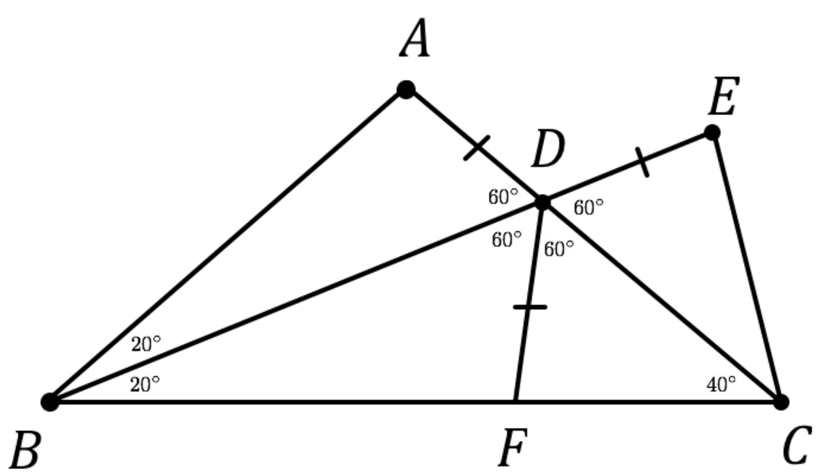


Answer: 80 degrees

Solution. Since $\angle ADB$ is an exterior angle of $\triangle DBC$, we have

$$\angle ADB = \angle DBC + \angle DCB = 20° + 40° = 60°.$$

Because *BD* bisects angle *B*, reflecting *A* in *BD* would take it to a point *F* on *BC* (see figure below).



AD would carry to *DF* and $\angle ADB$ to $\angle BDF$. Hence $FD = AD = DE$, and $\angle BDF = 60°$. Since *ADC* is a straight angle, $\angle FDC$ is also $60°$, and then, similarly, $\angle EDC$ is yet another $60°$ angle.

Hence, $\triangle DFC \equiv \triangle DEC$ (SAS), and $\angle E = \angle DFC = 180° - 60° - 40° = 80°$.

Problem 6.

Find the smallest positive integer k such that k^3 ends in the digits 111111 (in base 10).

Answer: 288,471

Solution: The only single-digit cube which ends in a 1 is 1. Next, if $(10a+1)^3 = 11 mod 100$, then expanding yields $30a + 1 \equiv 11 mod 100$, so $a = 7$. Then if $(100b + 71)^3 \equiv 111 mod 1000$, expanding yields $300b + 911 \equiv 111 mod 1000$, so $b = 4$. Continuing yields $(1000c + 471)^3 \equiv 1111 mod 10000$, so $3000c + 7111 \equiv 1111 mod 10000$ hence $c = 8$. Next, $(10000d + 8471)^3 \equiv 11111 mod 100000$, so $30000d + 71111 \equiv 11111 mod 100000$, hence $d = 8$. Finally, $(100000e + 88471)^3 \equiv 111111 mod 1000000 \equiv 300000e + 511111$, so $e \equiv 2 mod 10$. Hence the smallest positive k is 288471.

Problem 7.

Let $S = \{a, b, c, d, e, f\}$. Find the value of

$$\sum_{A\subseteq S}\sum_{B\subseteq S}|A\cap B|$$

where each sum runs over all subsets of S, and $|H|$ denotes the number of elements in the set H.

Answer: 6144

Solution: Let S be a set with n elements and suppose we choose each of A and B randomly from the 2^n subsets of S: the probability that any given element will be in the intersection is $\frac{1}{4}$, since the probability that it is in A is $\frac{1}{2}$ and (independently) the probability that it is in B is also $\frac{1}{2}$. Therefore, by the linearity of expectation, the expected number of elements in the intersection is $\frac{n}{4}$, so the total sum is $2^n \cdot 2^n \cdot \frac{n}{4} = n \cdot 4^{n-1}$. In the given problem $n = 6$, so the answer is $6 \cdot 4^5 = 6144$.

Problem 8.

Find the number of complex numbers z $(\neq 0,1)$ such that z, z^2, and z^4 form an equilateral triangle in the complex plane.

Answer: 4

Solution: Suppose the distinct complex numbers a, b, and c form an equilateral triangle. If we translate the coordinate system so that a is at the origin, then we have the three numbers $0, b-a, c-a$ that still form an equilateral triangle. If we think of $b-a$ and $c-a$ as vectors pointing outward from the origin, the triangle is equilateral precisely when one vector is given by rotating the other by $\frac{\pi}{3}$ radians. Since counterclockwiserotation by $\frac{\pi}{3}$ corresponds to multiplying by the complex number $e^{(i\pi/3)} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, we see that a, b, and c form an equilateral triangle if and only if $\frac{c-a}{b-a} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $\frac{c-a}{b-a} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. Thus we require $\frac{z^4 - z^2}{z - z^2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$; simplifying yields $-(z + z^2) = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Thus we see that there are 4 solutions for z, which are given approximately by $z \approx -1.0707 \pm 0.7587i \ and \ z \approx 0.0707 \pm 0.7587i$.

Note: One can solve explicitly for the four values of z, but the results are somewhat unpleasant. Explicitly, the solutions are given by $z = \frac{1}{24}\left(\pm 12 + \sqrt{78(1+\sqrt{13})} - \sqrt{6(1+\sqrt{3})} \pm 6i\sqrt{2(1+\sqrt{13})}\right)$, where the four choices of the $\pm$ are independent.