# Vermont Mathematics Talent Search, Solutions to Test 3, 2015-2016

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February 14, 2016

1. Find all pairs of positive integers $(x, y)$ such that $7x - 4y = 1$ and $y < x\sqrt{3}$.

   **Answer:** $(3,5)$, $(7,12)$, $(11,19)$.

   **Solution 1:** Reducing the relation $7x - 4y = 1$ modulo 4 yields $3x \equiv 1 \pmod 4$, so $x \equiv 3 \pmod 4$, meaning that $x = 3 + 4k$ for some $k \geq 0$. From this we immediately see that $y = 5 + 7k$, so we need only find the integers $k \geq 0$ satisfying the relation $(5 + 7k) < (3 + 4k)\sqrt{3}$. Squaring both sides yields $25 + 70k + 49k^2 < 27 + 72k + 48k^2$, or $(k-1)^2 < 3$. This clearly holds only for $k = 0, 1, 2$, so the solutions are $(x, y) = \boxed{(3,5),\ (7,12),\ \text{and}\ (11,19)}$.

   **Solution 2:** We have $y = \dfrac{7x-1}{4}$, so plugging into the second equation gives $\dfrac{7x-1}{4} < x\sqrt{3}$. Clearing the denominator and rearranging yields $(7 - 4\sqrt{3})x < 1$. Multiplying both sides by $7 + 4\sqrt{3}$ and observing that $(7-4\sqrt{3})(7+4\sqrt{3}) = 49 - 48 = 1$ yields $x < 7 + 4\sqrt{3} < 14$, so $1 \leq x \leq 13$. It is then a simple matter to plug in these 13 values of $x$ to see which ones give integral values for $y$. (Alternatively, we could reduce modulo 4 as in Solution 1 and notice that the desired $x$ will be those congruent to 3 modulo 4.) Either way, we obtain three solutions: $(x, y) = \boxed{(3,5),\ (7,12),\ \text{and}\ (11,19)}$.

2. A (nondegenerate) triangle with side lengths $\cos\theta$, $\cos\theta$, and $2\sin\theta$ has area $\cos 2\theta$. Find the area of the triangle whose side lengths are $\cos^2\theta$, $\cos^2\theta$, and $2\sin^2\theta$.

   **Answer:** $\sqrt{3}/9$.

   **Solution:** The original triangle is isosceles with base $2\sin\theta$ and slant height $\cos\theta$, so its height is $\sqrt{\cos^2\theta - \sin^2\theta} = \sqrt{\cos 2\theta}$. Then the triangle's area is equal to $\dfrac{1}{2}(2\sin\theta)(\sqrt{\cos 2\theta}) = \sin\theta\sqrt{\cos 2\theta}$. By hypothesis this value is equal to $\cos 2\theta$, so since $\cos 2\theta \neq 0$ (otherwise the triangle would be degenerate) we obtain $\sin\theta = \sqrt{\cos 2\theta}$. Squaring both sides yields $\sin^2\theta = 1 - 2\sin^2\theta$, so $\sin^2\theta = \dfrac{1}{3}$ and then $\cos^2\theta = \dfrac{2}{3}$. The side lengths of the second triangle are $\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{2}{3}$, so it is equilateral and its area is $\dfrac{(2/3)^2\sqrt{3}}{4} = \boxed{\dfrac{\sqrt{3}}{9}}$.

3. Let circle $O$ have radius 5 with diameter $\overline{AE}$. Point $F$ is outside circle $O$ such that lines $\overline{FA}$ and $\overline{FE}$ intersect circle $O$ at points $B$ and $D$, respectively. If $FA = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where $a, b, c$, and $d$ are rational. Find $a + b + c + d$.

   **Answer:** 15.

   **Solution:** Observe that $AE = AF$ so triangle $AEF$ is isosceles, and $\angle AEF = \angle AFE = 75°$. Now draw radius $\overline{OB}$: by the central angle property, $m\angle BOE = 60°$, and since $OB = OE = 5$ we see that $OBE$ is equilateral, so $BE = 5$. Since $\angle ABE$ is inscribed in a semicircle, it is right, so $AB^2 + BE^2 = AE^2$ whence $AB = 5\sqrt{3}$.

   Next draw radius $\overline{OD}$: since $OD = OE = 5$ and $\angle OED = 75°$, we see $\angle ODE = 75°$ and $\angle DOE = 30°$. Thus, $\triangle DOE$ is similar to $\triangle FAE$ with similarity ratio $\dfrac{1}{2}$, so $D$ is the midpoint of $FE$.

   By power-of-a-point we have $10(10 - 5\sqrt{3}) = FB \cdot FA = FD \cdot FE = 2(FD)^2$, so $FD = DE = \dfrac{1}{2}(5\sqrt{6} - 5\sqrt{2})$. Furthermore, since $\angle BOD = \angle BOE - \angle DOE = 30°$ we have $BD = DE = \dfrac{1}{2}(5\sqrt{6} - 5\sqrt{2})$ since they subtend the same angle.

Thus, the perimeter of quadrilateral $ABDE$ is $5\sqrt{3}+2\cdot\frac{1}{2}(5\sqrt{6}-5\sqrt{2})+10=10-5\sqrt{2}+5\sqrt{3}+5\sqrt{6}$, so $a+b+c+d=\boxed{15}$.

4. Right triangle $DEF$ has $E=(2,2)$, while $D$ lies on the curve $y=x^2-x$ and $F$ lies on the curve $y=3x-x^2$. If two vertices of $\triangle DEF$ have the same $x$-coordinate, find all possibilities for the area of $\triangle DEF$.

**Answer:** 1, 9, $6\sqrt{2}+8$, $6\sqrt{2}-8$.

**Solution:** Since $E$ lies on both curves, there are no other points on either curve with the same $x$-coordinate, so the points $D$ and $F$ must have the same $x$-coordinate.

If the right angle is not at $E$, since $DF$ is vertical we see that either $DE$ or $EF$ is horizontal. If $DE$ is horizontal, then $D$ is the other intersection of $y=2$ with $y=x^2-x$, so $D=(-1,2)$ and then $F=(-1,-4)$, for an area of $\frac{1}{2}\cdot 3\cdot 6=9$. If $EF$ is horizontal, then $F$ is the other intersection of $y=2$ with $y=3x-x^2$, so $F=(1,2)$ and then $D=(1,0)$, for an area of $\frac{1}{2}\cdot 1\cdot 2=1$.

If the right angle is at $E$, then $DE$ and $EF$ are perpendicular. If $D=(a,a^2-a)$ then $F=(a,3a-a^2)$, so $DE$ has slope $\frac{a^2-a-2}{a-2}=1+a$ and $EF$ has slope $\frac{3a-a^2-2}{a-2}=1-a$. These are perpendicular precisely when $(1+a)(1-a)=-1$; namely, when $a=\pm\sqrt{2}$. If $a=\sqrt{2}$, then $DF=4\sqrt{2}-4$ and the distance from $E$ to $DF$ is $2-\sqrt{2}$, so the area of the triangle is $\frac{1}{2}(4\sqrt{2}-4)(2-\sqrt{2})=6\sqrt{2}-8$. If $a=-\sqrt{2}$, then $DF=4\sqrt{2}+4$ and the distance from $E$ to $DF$ is $2+\sqrt{2}$, so the area of the triangle is $\frac{1}{2}(4\sqrt{2}+4)(2+\sqrt{2})=6\sqrt{2}+8$.

Thus, the area is $\boxed{1,\ 9,\ 6\sqrt{2}+8,\ \text{or } 6\sqrt{2}-8}$.

5. Find the number of integers $n$ such that the sum of all of the even divisors of $n$ is 2016.

**Answer:** 14.

**Solution:** Suppose $n$ has prime factorization $n=2^d p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$. Then the sum of the even divisors of $n$ is equal to $(2+2^2+\cdots+2^d)\cdot\prod_{i=1}^{k}(1+p_i+p_i^2+\cdots+p_i^{a_i})=2\cdot(2^d-1)\prod_{i=1}^{k}\frac{p_i^{a_i+1}-1}{p_i-1}$. After cancelling the factor of 2, we see that each term in the product must therefore be a divisor of $1008=2^4\cdot 3^2\cdot 7^1$. Since $2^d-1$ is odd (since $d>0$ as otherwise the product would be zero) and thus divides $3^2\cdot 7$, we see that $d$ must equal 1, 2, 3, or 6.

For $p$ odd, observe that $1+p+p^2>1008$ for $p\geq 32$, and that no term $1+p+p^2$ with $p$ prime divides 1008 for $p\leq 31$. Furthermore, $1+p+p^2+p^3>1008$ for $p\geq 11$, and these terms do not divide 1008 for $p=3$, 5, or 7. It is similarly easy to check that there are no terms of the form $\frac{p^{a+1}-1}{p-1}$ for $p=$3, 5, or 7 and $a>1$ that divide 1008. Hence, no odd prime appears to a power greater than 1 in the factorization of $n$.

A quick check shows that there are 13 odd primes $p$ such that $p+1$ divides 1008. They, with the factorization of $p+1$, are given in the table:

| $p$ | **3** | **5** | **7** | **11** | **13** | **17** | **23** | **41** | **71** | **83** | **167** | **251** | **503** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p+1$ | $2^2$ | $2\cdot 3$ | $2^3$ | $2^2\cdot 3$ | $2\cdot 7$ | $2\cdot 3^2$ | $2^3\cdot 3$ | $2\cdot 3\cdot 7$ | $2^3\cdot 3^2$ | $2^2\cdot 3\cdot 7$ | $2^3\cdot 3\cdot 7$ | $2^2\cdot 3^2\cdot 7$ | $2^3\cdot 3^2\cdot 7$ |

Now we break into cases based on the value of $d$:

- If $d=6$, then we need $\prod(1+p_i)=2^4$. However, the only terms that can appear in the product are $2^2$ and $2^3$, and so there is no way to get $2^4$.
- If $d=3$, then we need $\prod(1+p_i)=2^4\cdot 3^2$. We can do this with $(23,5)$ or $(17,7)$, yielding $n=920$ and $n=952$.
- If $d=2$, then we need $\prod(1+p_i)=2^4\cdot 3\cdot 7$. We can do this with $(83,3)$, $(41,7)$, $(23,13)$, or $(13,5,3)$, yielding $n=996$, 1148, 1196, and 780.

- If $d = 1$, then we need $\prod(1 + p_i) = 2^4 \cdot 3^2 \cdot 7$. We can do this with $(251, 3)$, $(167, 5)$, $(83, 11)$, $(71, 13)$, $(41, 23)$, $(41, 5, 3)$, $(17, 13, 3)$, or $(13, 11, 5)$, yielding $n = 1506$, 1670, 1826, 1846, 1886, 1230, 1326, and 1430.

In total, there are $\boxed{14}$ such integers $n$, namely: 780, 920, 952, 996, 1148, 1196, 1230, 1326, 1430, 1506, 1670, 1826, 1846, and 1886.

**Remark:** It turns out that for the even integers $1 \le n \le 2016$, the most common sum of even divisors is 1440, which appears 15 times. The second most common sum is 2016, which (as we saw above) appears 14 times.

6. Suppose $a$, $b$, and $c$ are positive real numbers such that $\max(a, b, c) \le 4 \min(a, b, c)$. Prove that $2ab + 2ac + 2bc \ge a^2 + b^2 + c^2$, and determine when equality can occur.

**Answer:** Equality can occur precisely when $(a, b, c) = (t, t, 4t)$, $(t, 4t, t)$, or $(4t, t, t)$ for some $t > 0$.

**Solution 1:** Since everything in the problem is invariant under scaling and reordering the variables, we may assume that $a \le b \le c$ and that $a = 1$. Then the problem becomes: if $1 \le b \le c \le 4$, show that $2b + 2c + 2bc \ge 1 + b^2 + c^2$. Completing the square on the right gives $2b + 2c \ge 1 + (c - b)^2$, and we may rearrange this further to give $4b \ge (c - b - 1)^2$. Now since $b \ge 1$ and $-1 \le c - b - 1 \le 2$, we see that $4b \ge 4$ whereas $(c - b - 1)^2 \le 4$. So the desired inequality $4b \ge (c - b - 1)^2$ is always true, and equality can only hold when both terms are equal to 4: that is, when $b = 1$ and $b - c - 1 = 2$, or equivalently, when $(b, c) = (1, 4)$. Therefore, equality in the original problem can hold if and only if $(a, b, c) = (t, t, 4t)$, $(t, 4t, t)$, or $(4t, t, t)$ for some $t > 0$.

**Solution 2:** As in Solution 1, we may reorder the variables and rescale to assume that $a = 1$ and that $1 \le b, c \le 4$. We wish to show that the maximum value of the function $f(b, c) = 2b + 2c + 2bc - 1 - b^2 - c^2$ for $1 \le b, c \le 4$ is equal to zero. If we think of this function as a function of $b$ only, with $f(b, c) = -b^2 + (2c + 2)b + (2c - 1 - c^2)$ is a parabola opening downward, so its maximum value on the interval $[1, 4]$ either occurs at an endpoint or at the vertex located at $b = c + 1$. Similarly, as a function of $c$, the maximum value of $h(c) = -c^2 + (2b + 2)c + (2b - 1 - b^2)$ either occurs at $c = 1$, $c = 4$, or $c = b + 1$. There are nine possible configurations: $(b, c) = (1, 1)$, $(1, 4)$, $(1, b + 1)$, $(4, 1)$, $(4, 4)$, $(4, b + 1)$, $(c + 1, 1)$, $(c + 1, 4)$, $(c + 1, b + 1)$. These respectively give the possible points $(b, c) = (1, 1)$, $(1, 4)$, $(1, 2)$, $(4, 1)$, $(4, 4)$, $(4, 5)$ [not allowed], $(2, 1)$, and $(5, 4)$ [not allowed], with the last case being contradictory. Plugging in the six possible points shows that the maximum value of $f(b, c)$ is 0, occurring at $(b, c) = (1, 4)$ and $(4, 1)$. This proves the required inequality and gives the equality cases as in Solution 1.

**Solution 3:** Observe that $\sqrt{a} + \sqrt{b} \ge 2\sqrt{\min(a, b, c)} = \sqrt{4\min(a, b, c)} \ge \sqrt{\max(a, b, c)} \ge \sqrt{c}$, so $\sqrt{a} + \sqrt{b} \ge \sqrt{c}$ with equality if and only if $b = a$ and $4a = c$.

Similarly, $\sqrt{a} + \sqrt{c} \ge \sqrt{b}$ with equality if and only if $a = c$ and $4a = b$, and $\sqrt{b} + \sqrt{c} \ge \sqrt{a}$ with equality if and only if $b = c$ and $4b = a$.

Then

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c}) \ge 0$$

since each term is positive. Multiplying out the right-hand side yields, after some algebra,

$$2ab + 2ac + 2bc - a^2 - b^2 - c^2 \ge 0$$

which is equivalent to the desired inequality. Equality can hold only when one of the terms in the product is equal to zero, which happens precisely when $a = b = c/4$, $a = c = b/4$, or $b = c = a/4$.

**Remark:** It is also possible to solve this problem using calculus. With the notation of Solution 2, the key point is to observe that there are no critical points of the function $f(b, c)$ inside the square $[1, 4] \times [1, 4]$, as both partial derivatives cannot be equal to zero simultaneously: this would require $2 + 2b - 2c = 2 - 2b + 2c = 0$ which has no solutions. Therefore, any minimum or maximum must therefore occur on the boundary of the square, meaning that at least one of $b, c$ must be equal to 1 or 4. This reduces the problem to a one-variable maximization of a quadratic function whose solutions are straightforward to find.