Vermont Mathematics Talent Search, Solutions to Test 2, 2016-2017

Test and Solutions by Jean Ohlson and Evan Dummit

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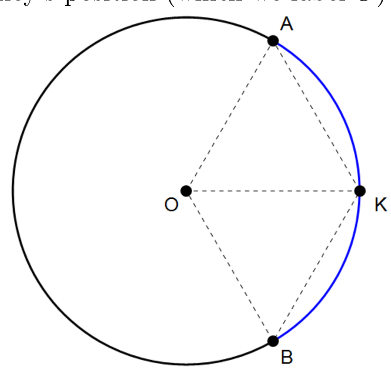
1. Three students on Vermont's A team at the American Regional Math League competition are standing in a large open field. Matthew and Kailey are standing in the same spot and Kevin is standing 12 meters away. Matthew chooses a random direction and walks a distance of 12 meters in this direction away from Kailey. What is the probability that Matthew is closer to Kevin than Kailey is to Kevin?

   Answer: 1/3.

   Solution: After moving, Matthew will be located at a random (uniformly chosen) point along the circumference of the circle centered at Kailey's position (which we label O) of radius 12 meters.

   Let A and B be the two positions on the circle O where Matthew, Kailey, and Kevin are all 12 meters away from each other. Then the points on the circle which place Matthew closer to Kevin than Kailey is to Kevin are those lying on the arc AB that passes through Kevin's position, which (per the diagram) is the shorter arc from A to B. Since $OK = 12$, we see that $\triangle AOK$ and $\triangle BOK$ are both equilateral, and so $m\angle AOB = 120°$. Thus, the desired arc measures one-third of the total circumference of the circle, so the probability that Matthew ends up on this arc if he walks in a random direction is $\boxed{1/3}$.

2. Erik has a rectangular piece of cardboard measuring x units by y units. He cuts out four identically-oriented rectangles (with sides parallel to the full piece of cardboard) of dimensions 2 units by 4 units from the four corners of his piece of cardboard, in such a way that the resulting material can be folded into a covered rectangular box (including the top) whose volume is 72 cubic units. Find the area of Erik's original piece of cardboard.

   Answer: 156 square units.

   Solution: Observe that the cut box consists of a central $(x-4) \times (y-8)$ piece, bordered by two $(x-4) \times 4$ pieces and two $(y-8) \times 2$ pieces around its perimeter. The $(x-4) \times 4$ pieces must then fold over to cover the top: they must be creased at height 2, leaving two identical $(x-4) \times 2$ pieces to cover the top of the box, which therefore has dimensions $(x-4) \times 4$. Since the bottom of the box has dimensions $(x-4) \times (y-8)$, we must have $y = 12$. Now, since the height of the box is 2 units, the volume condition implies $2(x-4)(y-8) = 72$, so that $x - 4 = 9$ and thus $x = 13$. The original piece of cardboard therefore has area $12 \cdot 13 = \boxed{156}$ square units.

3. The positive integer N is divisible by every two-digit integer. If N is written as the product of k two-digit integers, find the smallest possible value of k.

Answer: 22.

Solution: Assume for the moment that N is as small as possible, so that N is the least common multiple of the set of 2-digit integers. Note that the prime factorization of any two-digit integer can contain at most 6 factors of 2 (since $2^7 = 128$ is too large), 4 factors of 3 ($3^5 = 243$), 2 factors of 5 ($5^3 = 125$), 2 factors of 7 ($7^3 = 343$), and 1 factor of any prime larger than 10. Thus, the least common multiple N is $N = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot \cdots \cdot 97$, where the omitted part of the product contains all of the 2-digit primes (of which there are 21).

Each 2-digit prime must be in its own of the k factors, so $k \geq 21$. If it were possible to write N as a product of exactly 21 factors, then 7^2, 5^2, and 3^4 would all need to be absorbed into the product $11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot \cdots \cdot 97$ without making any factor exceed 99. The only possibility would be that one 7 goes with each of 11 and 13, and one 5 goes with each of 17 and 19: but then only 23, 29, and 31 can absorb a 3 without exceeding 99, but there are four 3s. Thus $k \geq 22$. We can see that $k = 22$ is possible by extending the previous analysis and writing $N = (2^5 \cdot 3) \cdot (7 \cdot 11) \cdot (7 \cdot 13) \cdot (5 \cdot 17) \cdot (5 \cdot 19) \cdot (3 \cdot 23) \cdot (3 \cdot 29) \cdot (3 \cdot 31) \cdot (2 \cdot 37) \cdot 41 \cdot \cdots \cdot 97$, and there are 22 terms each of which is a two-digit integer. Finally, if we chose a larger N, then we certainly could not write it as a product of fewer terms. Thus, we conclude $k = \boxed{22}$ is the smallest possible.

4. Let $a = \log_3 5$, $b = 2\log_2 a$, $c = 9^b$, $d = \log_a(1/c)$, $e = 3ad$, $f = 5^{4/e}$, and $g = \log_{\sqrt{2}}(2/f^3)$. Compute the value of g, in simplest form.

Answer: 4.

Solution: We compute, successively, the following:

$$\begin{aligned} a &= \log_3 5 \\ b &= 2\log_2(\log_3 5) \\ c &= 9^{2\log_2(\log_3 5)} = 3^{4\log_2(\log_3 5)} \\ d &= -\log_{\log_3 5}(3^{4\log_2(\log_3 5)}) = -4\log_2(\log_3 5) \cdot \log_{\log_3 5} 3 = -4\log_2 3 \end{aligned}$$

using the change of base formula $\log_x y \cdot \log_y z = \log_x z$ with $x = 2$, $y = \log_3 5$, and $z = 3$. We obtain

$$e = -12\log_3 5 \cdot \log_2 3 = -12\log_2 5$$

again using the change of base formula. Then

$$f = 5^{4/(-12\log_2 5)} = 5^{-(\log_5 2)/3} = 5^{\log_5 2^{-1/3}} = 2^{-1/3}$$

upon writing $1/\log_2 5 = \log_5 2$. Finally, we obtain

$$g = \log_{\sqrt{2}}(2/2^{-1}) = \log_{\sqrt{2}} 4 = \log_{\sqrt{2}}(\sqrt{2})^4 = \boxed{4}.$$

5. For positive real numbers x, y, and z, let $f(x,y,z) = \min\left(\frac{2}{x}, \frac{2z}{y}, x^2 + 2xy, \frac{x}{z}\right)$. Find, with proof, the maximum value of $f(x,y,z)$ and all triples (x,y,z) at which this maximum is achieved.

Answer: The maximum is 2, achieved uniquely at the triple $(x,y,z) = (1, 1/2, 1/2)$.

Motivation: Since there are three variables and four quantities that depend on them, it is a reasonable guess that the maximum will occur when all four terms are equal (otherwise, there should be some way to change the values of some of the variables slightly to make the smallest of the terms a bit bigger). Working with this guess, we obtain the system $\frac{2}{x} = \frac{2z}{y} = x^2 + 2xy = \frac{x}{z}$ from which we obtain $z = x^2/2$, $y = xz = x^3/2$, and then $2/x = x^2 + x^4$. The resulting equation $2 = x^3 + x^5$ has a unique real solution $x = 1$ (note that the function $f(x) = x^3 + x^5$ is monotone increasing), so we guess that the maximum occurs with $x = 1$ and $y = z = 1/2$.

Solution: Observe first that if $x = 1$ and $y = z = 1/2$, then each of $\frac{2}{x}$, $\frac{2z}{y}$, $x^2 + 2xy$, and $\frac{x}{z}$ is equal to 2, so the maximum of f is at least 2. Now suppose the maximum were some value $\alpha \geq 2$: then we would have $x^2 + 2xy \geq \alpha$, $\frac{2}{x} \geq \alpha$, $\frac{2z}{y} \geq \alpha$, and $\frac{x}{z} \geq \alpha$. Multiplying the last 3 relations yields $\frac{4}{y} \geq \alpha^3$, so $y \leq \frac{4}{\alpha^3}$. Likewise, since $\frac{2}{x} \geq \alpha$ we see that $x \leq \frac{2}{\alpha}$. We then can write

$$\alpha \leq x^2 + 2xy \leq \frac{4}{\alpha^2} + \frac{16}{\alpha^4}$$

from which we obtain $\alpha^5 - 4\alpha^2 \leq 16$. However, if $\alpha > 2$ then $\alpha^5 - 4\alpha^2 = \alpha^2(\alpha^3 - 4) > 2^2 \cdot (2^3 - 4) = 16$, which is a contradiction. Therefore, f cannot take any value larger than 2, so the maximum is 2.

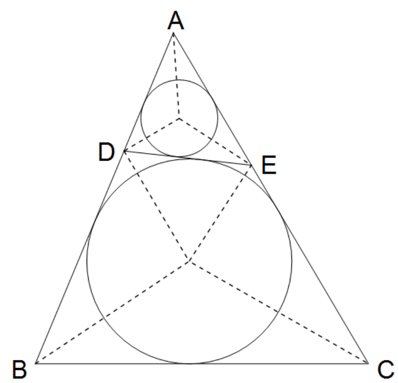
Furthermore, from our analysis, equality can hold only when each of $\frac{2}{x}$, $\frac{2z}{y}$, $x^2 + 2xy$, and $\frac{x}{z}$ is equal to 2. This immediately yields $x = 1$, $z = 1/2$, and $y = 1/2$: since $f(1, 1/2, 1/2) = 2$, we see that this is the unique place f attains its maximum.

Remark: If x, y, z are allowed to take negative values, the maximum of f is unchanged, because if any of x, y, z is negative, then at least one of $2/x$, x/z, $2z/y$ will also be negative.

6. In triangle ABC, the point D on AB and the point E on AC are chosen such that a circle of radius 8 can be inscribed in quadrilateral $BCDE$ and a circle of radius 3 can be inscribed in triangle ADE. If $BC = 26$ and $BD + CE = 36$, find the perimeter of $\triangle ABC$.

Answer: 84.

Solution 1: Refer to the diagram below:



Let $AC = b$ and $AB = c$, and also let brackets denote areas, so that $[ABC]$ denotes the area of ABC. Observe that $BC + DE = BD + CE$ since $BCDE$ has a circle inscribed in it, so $DE = 10$. Also notice that $AD = b - BD$ and $AE = c - CE$.

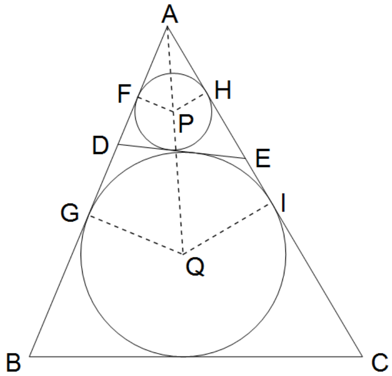
Drawing segments from the vertices to $BCDE$ to the center of the inscribed circle, as pictured above, divides $BCDE$ into four triangles each with height 8 and base equal to one side of $BCDE$. Thus, $[BCDE] = \frac{1}{2} \cdot 8 \cdot (BC + CD + DE + BE) = 4 \cdot (2 \cdot 36) = 288$.

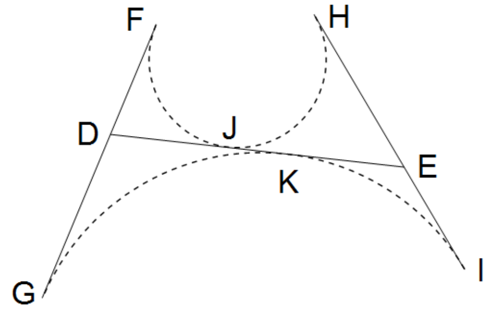
In a similar way, by drawing segments from the vertices of ADE to the center of its inscribed circle, we see that $[ADE] = \frac{1}{2} \cdot 3 \cdot (AD + AE + DE) = \frac{1}{2} \cdot 3 \cdot (b + c - 36 + 10) = \frac{3}{2}(b + c) - 39$.

Note that the circle inscribed in $BCDE$ is the incircle of triangle ABC, so by the same argument as with ADE, we have $[ABC] = \frac{1}{2} \cdot 8 \cdot (b + c + 26) = 4(b + c) + 104$.

Since $[ABC] = [ADE] + [BCDE]$, we see that $4(b+c)+104 = \frac{3}{2}(b+c) - 39 + 288$. Therefore, $b+c = 58$ and the perimeter of $\triangle ABC$ is $26 + b + c = \boxed{84}$.

Solution 2: Refer to the diagrams below:





Let P be the center of the smaller circle and Q be the center of the larger one. Also take F and G to be the tangency points of AB with circles P and Q respectively, and H and I to be the tangency points of AC with circles P and Q respectively.

Observe as in Solution 1 that $DE = 10$, and let $AF = x$.

Then line APQ is the bisector of angle A, so $\triangle AFP$ and $\triangle AGQ$ are similar (right) triangles. Thus, since $FP = 3$ and $GQ = 8$, we have $AG = \frac{8}{3}x$, so $DF + DG = \frac{5}{3}x$.

In the same way, we see that $\triangle AHP$ and $\triangle AIQ$ are similar, so $EH + EI = \frac{5}{3}x$.

Now let J be the point of tangency of circle P with DE and K be the point of tangency of circle Q with DE. Since common external tangents have the same length, we see that $JK = DK - DJ = DG - DF$, and, symmetrically, $JK = EJ - EK = EH - EI$.

Thus, $DG - DF = EH - EI$ and $DG + DF = EH + EI$, so $DG = EH$ and $DF = EI$. Then, $\frac{5}{3}x = EH + EI = EJ + DJ = DE = 10$, so $x = 6$.

Then, again using common external tangents, $BG + CI = BC = 26$, so the perimeter of $\triangle ABC$ is $AG + BG + BC + CI + AI = 52 + \frac{16}{3}x = \boxed{84}$.

Remark: One triangle (not the only one!) satisfying all of the given conditions has $AD = 10$, $DB = 18$, $AE = 12$, $EC = 18$, and $DE = 10$.