Vermont Mathematics Talent Search, Solutions to Test 2, 2018-2019

Test and Solutions by Kiran MacCormick and Evan Dummit

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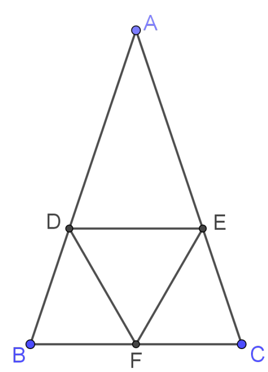
1. In a game in the National Fair-Coin-Flipping League, each flip counts for one point, and a team must score 7 points to win. If the Heads currently have 5 points and the Tails currently have 2 points, what is the probability that the Heads will win this game?

   Answer: 57/64.

   Solution: We find the probability that the Tails win the game, and then subtract from 1. Since the Heads already have 5 points, the only way the Tails can win is 7-5 or 7-6. If they win 7-5, then they must flip 5 tails in a row, which occurs with probability 1/32. If they win 7-6, then they must flip 5 tails and 1 heads, where the heads is not last (since the Tails would already have won 7-5 before that flip occurred): this occurs with probability 5/64. Therefore, the probability that the Tails win is $1/32 + 5/64 = 7/64$, so the probability that the Heads win is $1 - 7/64 = \boxed{57/64}$.

2. In $\triangle ABC$, $AB = AC = 8$ and $BC = 4$. Points D, E and F are located on sides AB, AC, and BC (respectively) such that DE is parallel to BC and $\triangle DEF$ is equilateral. Find the perimeter of $\triangle DEF$.

   Answer: $15 - 3\sqrt{5}$.



   Solution 1: By symmetry, notice that $BF = CF = 2$, and since AF is the altitude to side BC, by the Pythagorean theorem we see that $AF = \sqrt{8^2 - 2^2} = 2\sqrt{15}$. Now let $DE = x$. Since DE is parallel to BC (again by symmetry) we see that $\triangle ADE$ is similar to $\triangle ABC$, and so by similarity we see that the height of $\triangle ADE$ is $x\sqrt{15}/2$. Furthermore, since $\triangle DEF$ is equilateral, its height is $x\sqrt{3}/2$. Since the sum of the heights of $\triangle ADE$ and $\triangle DEF$ is the height of $\triangle ABC$, we obtain $x\sqrt{15}/2 + x\sqrt{3}/2 = 2\sqrt{15}$. Solving for x yields

$$x = \frac{2\sqrt{15}}{\sqrt{15}/2 + \sqrt{3}/2} = \frac{2\sqrt{15}}{\sqrt{15}/2 + \sqrt{3}/2} \cdot \frac{\sqrt{15}/2 - \sqrt{3}/2}{\sqrt{15}/2 - \sqrt{3}/2} = \frac{15 - 3\sqrt{5}}{3} = 5 - \sqrt{5}.$$

   The perimeter of $\triangle DEF$ is then $3x = \boxed{15 - 3\sqrt{5}}$.

   Solution 2: Let $DE = x$. Observe that $\triangle ADE$ is similar to $\triangle ABC$, so $AD = AE = 2x$ and thus $BD = 8-2x$. Since AF is the altitude to side BC, we see $BF = 2$ and $\cos\angle ABF = \frac{1}{4}$. Since $DF = x$, applying the law of cosines in triangle DBF yields $(8-2x)^2 + 2^2 - 2 \cdot (8-2x) \cdot 2 \cdot \frac{1}{4} = x^2$. Expanding and rearranging yields $3x^2 - 30x + 60 = 0$, or $x^2 - 10x + 15 = 0$, whose solution is $x = 5 \pm \sqrt{5}$. Since $x < 4$ we must have the minus sign, so $x = 5 - \sqrt{5}$ and the desired perimeter is $3x = \boxed{15 - 3\sqrt{5}}$.

3. Let C be the circle $x^2 + y^2 = 87$, and recall that a lattice point is a point whose coordinates are both integers. We say a lattice point is "lattice-closest" to C if no other lattice point has a smaller distance to C. The set of lattice-closest points to C forms a polygon: determine the area of this polygon.

Answer: 238.

Solution: The shortest distance from a point (a, b) to the circle $x^2 + y^2 = 87$ lies along the line along the line segment joining (a, b) to $(0, 0)$, and is equal to $|\sqrt{a^2 + b^2} - \sqrt{87}|$, the positive difference between $\sqrt{a^2 + b^2}$ and the radius $\sqrt{87}$ of the circle. The minimum value of $|\sqrt{a^2 + b^2} - \sqrt{87}|$ clearly occurs when $a^2 + b^2$ is as close as possible to 87. It is straightforward to verify that 86, 87, and 88 are not the sum of two squares, but $85 = 2^2 + 9^2$ and $89 = 5^2 + 8^2$ are. Since $\sqrt{89} - \sqrt{87} = \dfrac{2}{\sqrt{89} + \sqrt{87}} < \dfrac{2}{\sqrt{87} + \sqrt{85}} = |\sqrt{85} - \sqrt{87}|$, we see that the minimum distance is $\sqrt{89} - \sqrt{87}$.

Therefore, the desired polygon has vertices at those points (a, b) where $a^2 + b^2 = 89$. These points are $(\pm 5, \pm 8)$ and $(\pm 8, \pm 5)$, so the polygon is an octagon, given by removing four isosceles right triangles of side length 3 from the corners of the square whose vertices are $(\pm 8, \pm 8)$. The area of the polygon is therefore $16^2 - 4 \cdot 9/2 = \boxed{238}$.

4. If x, y, z are real or complex numbers such that $xyz = 7$ and $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = x^3 + y^3 + z^3 = 1$, find all possible values for $x + y + z$.

Answer: 1, 4, −5.

Solution: Multiply $xyz = 7$ and $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$ to obtain $xy + xz + yz = 7$. Also, since $x^3 + y^3 + z^3 = (x + y + z)^3 - 3(xy + xz + yz)(x + y + z) + 3(xyz)$, if we let $s = x + y + z$ we see that s satisfies the equation $1 = s^3 - 21s + 21$, so that $s^3 - 21s + 20 = 0$. Factoring yields $(s - 1)(s - 4)(s + 5) = 0$ so $s = \boxed{1, 4, -5}$. Each of these possibilities can occur, because x, y, z are the roots of the polynomial $p(t) = (t - x)(t - y)(t - z) = t^3 - (x + y + z)t + (xy + xz + yz)t - (xyz) = t^3 - st + 7t - 7$.

5. Find the sum of all values of θ, $0 \le \theta \le 2\pi$, for which $(\sin\theta + i\cos\theta)^{15} = \cos\theta + i\sin\theta$.

Answer: $33\pi/2$.

Solution 1: Since $-i(\sin\theta + i\cos\theta) = \cos\theta - i\sin\theta$, we see that $\sin\theta + i\cos\theta = i(\cos(-\theta) + i\sin(-\theta)) = e^{i\pi/2}e^{-i\theta}$. Then we can write

$$e^{i\theta} = \cos\theta + i\sin\theta = (\sin\theta + i\cos\theta)^{15} = (e^{i\pi/2}e^{-i\theta})^{15} = e^{15i\pi/2}e^{-15i\theta}$$

from which we see $e^{16i\theta} = e^{15i\pi/2} = e^{3i\pi/2}$. Therefore, $16i\theta = \dfrac{3i\pi}{2} + 2\pi ik$ for some integer k, meaning that $\theta = \dfrac{3\pi}{32} + \dfrac{\pi k}{8}$ for some integer k. We thus obtain the 16 solutions $\theta = \dfrac{3\pi}{32}, \dfrac{7\pi}{32}, \cdots, \dfrac{63\pi}{32}$, and now using the formula for the sum of an arithmetic progression yields that the desired sum is $\dfrac{1}{2} \cdot 16 \cdot \dfrac{66\pi}{32} = \boxed{\dfrac{33\pi}{2}}$.

Solution 2: Since $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta) = i(\cos(-\theta) + i\sin(-\theta))$, by de Moivre's Theorem, we see that

$$(\sin\theta + i\cos\theta)^{15} = i^{15} \cdot (\cos(-\theta) + i\sin(-\theta))^{15} = i^{15} \cdot (\cos(-15\theta) + i\sin(-15\theta)) = \sin(-15\theta) - i\cos(-15\theta)$$

and so equating real and imaginary parts yields $\cos(\theta) = -\sin(15\theta)$ and $\sin(\theta) = -\cos(15\theta)$, which is equivalent to $\sin(3\pi/2 - \theta) = \sin(15\theta)$ and $\cos(3\pi/2 - \theta) = \cos(15\theta)$. These two conditions together are equivalent to saying that $\dfrac{3\pi}{2} - \theta$ and 15θ differ by a multiple of 2π, which is to say $15\theta = \dfrac{3\pi}{2} - \theta + 2\pi k$ for some integer k. As in Solution 1, we obtain the 16 solutions $\theta = \dfrac{3\pi}{32}, \dfrac{7\pi}{32}, \cdots, \dfrac{63\pi}{32}$, with sum $\boxed{\dfrac{33\pi}{2}}$.

Remark: Solution 2 is really the same as Solution 1, just written out explicitly in terms of real and imaginary parts rather than using complex exponentials.

6. Evan has a large supply of square tiles of side length 1, 2, 4, 8, 16, ... , 1024. Determine, with proof, the minimum number of tiles Evan needs to cover a 2019×2019 region completely with tiles such that no tiles overlap.

Answer: 6288 tiles.

Solution 1: A general recursive method for tiling an $n \times n$ square is as follows: if n is odd, tile the bottom and right edge with $(2n-1)$ 1×1 tiles, leaving an $n-1$ by $n-1$ square. If n is even, take a tiling for an $n/2$ by $n/2$ square and double the size of all the tiles. When applied to a 2019×2019 square, this method produces a tiling with the following tiles: 4037 (1×1) + 2017 (2×2) + 125 (32×32) + 61 (64×64) + 29 (128×128) + 13 (256×256) + 5 (512×512) + 1 (1024×1024), for a total of 6288.

It remains to prove that this tiling is optimal. We will show recursively that at least 4037 1×1 tiles, 2017 additional 1×1 or 2×2 tiles, and so forth, are required for such a tiling.

First, color black every square whose coordinates are both even, and color white the remaining squares. Observe that any 2×2 or larger tile covers 3 times as many white squares as black squares. Since there are 1009^2 black squares and $3 \cdot 1009^2 + 4037$ white squares, there must be at least 4037 1×1 tiles to cover the "extra" white squares.

Next, color blue every square whose coordinates are both multiples of 4, and color white the remaining squares. Observe that any 4×4 or larger tile covers 15 times as many white squares as blue squares. Since there are 504^2 blue squares and $15 \cdot 504^2 + 12105$ white squares, the excess white squares must be covered by 1×1 or 2×2 tiles. Removing the 4037 squares covered by 1×1 tiles from the previous step, there are 8068 remaining squares to be covered by 1×1 or 2×2 tiles, which requires at least $8068/4 = 2017$ additional tiles.

We can repeat this argument for the remaining colorings (of squares whose coordinates are both multiples of 8, 16, 32, ... , 1024) to obtain a lower bound of $4037+2017+0+0+0+125+61+29+13+5+1 = 6288$ tiles. Therefore, the minimum is indeed $\boxed{6288}$.

Solution 2: As in the first solution we observe that there is a tiling using 6288 tiles. To show that this number is minimal, we first establish the following lemma:

Lemma: The maximum possible number of $n \times n$ tiles that will fit inside an $a \times b$ rectangle (with sides parallel to the sides of the large rectangle, along the grid lines) is $\lfloor n/a \rfloor \cdot \lfloor n/b \rfloor$.

Proof: It is clearly possible to achieve $\lfloor n/a \rfloor \cdot \lfloor n/b \rfloor$ tiles by packing the tiles into the upper left corner of the rectangle. Now suppose we have an arrangement with a maximal possible number of tiles, and consider the upper left $n \times n$ region of the rectangle. If none of the squares is occupied, then we could add another $n \times n$ tile here, so at least one square is occupied by a tile. Consider the upper-left corner of this tile: if it is not the upper left square of the large rectangle, then (since no other tile could be between this tile and the left or top edge of the rectangle) we may slide this tile up and to the left to align it in the upper-left corner of the large rectangle. Now remove this $n \times n$ region from the rectangle and repeat the argument iteratively, in each of the $\lfloor n/a \rfloor \cdot \lfloor n/b \rfloor$ total $n \times n$ regions, in "reading order": after each move, we do not decrease the total number of $n \times n$ tiles in the large rectangle, and after we are done, they are packed tightly into the upper left corner. It is then immediate that no additional tiles can be added to the construction, so $\lfloor n/a \rfloor \cdot \lfloor n/b \rfloor$ is the maximal possible number of tiles.

Now we recursively apply the lemma: with $a = b = 2019$ and $n = 1024$, we see that there is at most $\lfloor 2019/1024 \rfloor^2 = 1$ tile of size 1024×1024.

Now imagine splitting this 1024×1024 tile into 4 tiles of size 512: with $a = b = 2019$ and $n = 512$, we see that there are at most $\lfloor 2019/512 \rfloor^2 = 9$ tiles of size 512, and therefore there can be at most $9 - 4 = 5$ additional tiles of size 512.

Similarly, if we then split these 9 512-tiles into 36 256-tiles, applying the lemma with $a = b = 2019$ and $n = 256$, we see there are at most $\lfloor 2019/256 \rfloor^2 = 49$ tiles of size 256, meaning that there can be at most $49 - 36 = 11$ additional tiles of size 256.

By continuing this argument down to the smallest tiles, we deduce that a minimum of 1 (1024×1024) + 5 (512×512) + 13 (256×256) + 29 (128×128) + 61 (64×64) + 125 (32×32) + 2017 (2×2) + 4037 (1×1) tiles are needed, which is $\boxed{6288}$ tiles in total.

Remark: Solutions 1 and 2 yield the same tiling algorithm, merely described differently (Solution 1 begins with the smallest tiles, while Solution 2 begins with the largest ones), and the coloring argument from Solution 1 yields another proof of the Lemma from Solution 2. The procedure in both solutions yields an optimal tiling for any collection of tile sizes where each size divides the next larger one.