

Vermont Mathematics Talent Search, Solutions to Test 3, 2018-2019

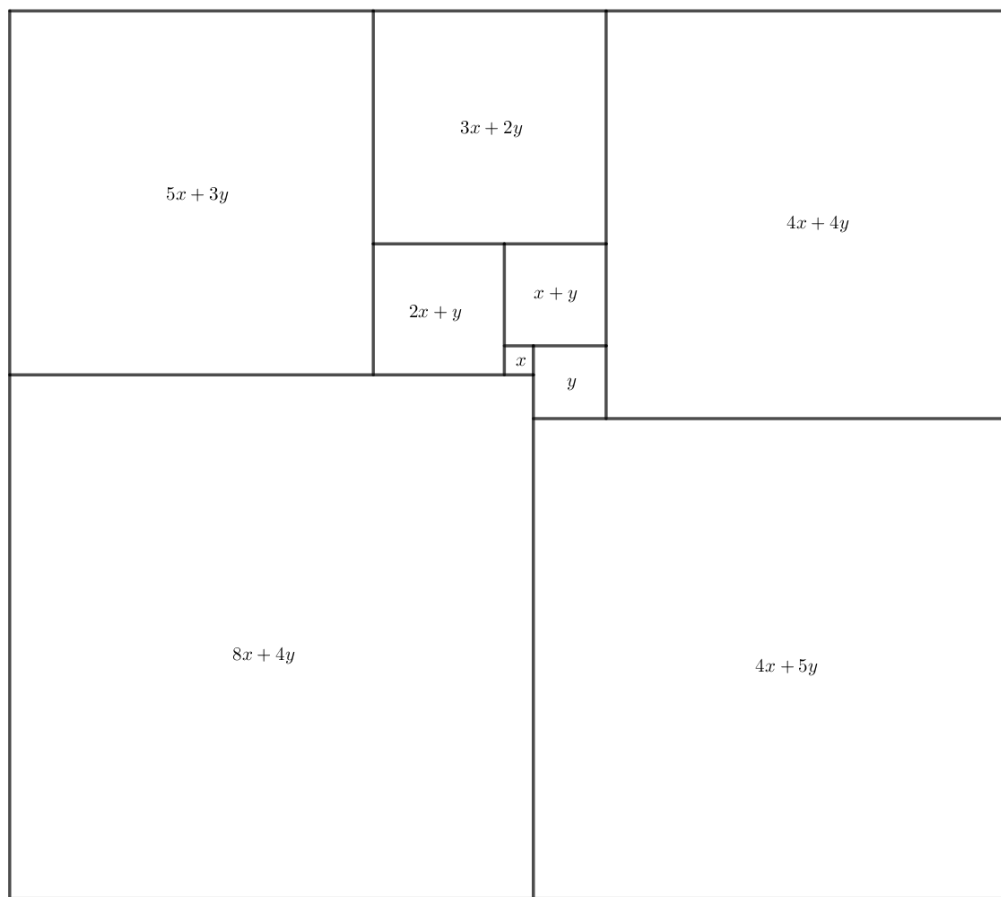
Test and Solutions by Kiran MacCormick and Evan Dummit

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1. The diagram at right shows nine non-overlapping squares assembled to form a rectangle. The length and width of the rectangle are relatively prime positive integers. Find the perimeter of the rectangle.

Answer: 260.

Solution: If the side length of the smallest square is x and the next-smallest square is y , we can fill in all the side lengths of the remaining squares by comparing to those we have already labeled, as follows:



Since the left and right sides of the rectangle have the same length, this yields the relation $(5x + 3y) + (8x + 4y) = (4x + 4y) + (4x + 5y)$, so that $5x = 2y$. To obtain the smallest possible integer side lengths we therefore want to take $x = 2$ and $y = 5$. Then the large rectangle has side lengths $13x + 7y = 61$ and $12x + 9y = 69$. Since 61 and 69 are relatively prime, these are the desired side lengths, yielding a total perimeter of $2(61 + 69) = \boxed{260}$.

2. The sequence of digits 1234567891011121314151617181920... is obtained by writing the positive integers in order. If the 10^n th digit in this sequence occurs in the part of the sequence in which the m -digit numbers are placed, define $f(n)$ to be m . For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit number 55. Find $f(2019)$.

Answer: 2016.

Solution: There are 9 one-digit numbers, 90 two-digit numbers, 900 three-digit numbers, and in general, there are $9 \cdot 10^{k-1}$ numbers with exactly k digits. Together, the k -digit numbers contribute a total of $k \cdot 9 \cdot 10^{k-1}$ digits to the sequence, and so the total length of the sequence after we have finished writing all the m -digit numbers is $a_m = \sum_{k=1}^m 9k \cdot 10^{k-1}$. Then $f(n)$ is the integer m with $a_{m-1} < 10^n \leq a_m$. We have

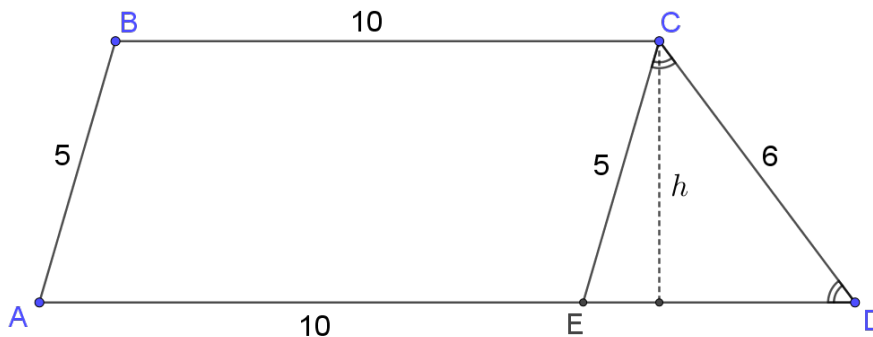
$a_m/9 = 1 + 2 \cdot 10 + 3 \cdot 10^2 + \dots + m \cdot 10^{m-1}$ and $10a_m/9 = 10 + 2 \cdot 10^2 + 3 \cdot 10^3 + \dots + m \cdot 10^m$, so subtracting yields $a_m = m \cdot 10^m - (1 + 10 + 10^2 + \dots + 10^{m-1}) = m \cdot 10^m - \frac{10^m - 1}{9} = (m - \frac{1}{9})10^m + \frac{1}{9} \approx m \cdot 10^m$.

In particular, because $10^3 < 2019 < 10^4$ we can see that $a_{2015} < 10^4 \cdot 10^{2015} = 10^{2019}$ while $a_{2016} > 10^3 \cdot 10^{2016} = 10^{2019}$. Therefore, $a_{2015} < 10^{2019} < a_{2016}$, so the answer is $\boxed{2016}$.

3. In trapezoid $ABCD$, AD is parallel to BC and $m\angle D = m\angle C - m\angle A$. If $AB = 5$, $BC = 10$, and $CD = 6$, find the area of trapezoid $ABCD$.

Answer: 60.

Solution: Let E be on AD such that $m\angle BCE = m\angle A$: then $m\angle DCE = m\angle D$, so $ABCE$ is a parallelogram and $\triangle CDE$ is isosceles.



Then $AB = CE = DE = 5$ and $BC = AE = 10$. If the height of the trapezoid is h , then dropping the altitude from C to AD and using the Pythagorean Theorem in the resulting triangles yields $5 = \sqrt{25 - h^2} + \sqrt{36 - h^2}$, or equivalently, $5 - \sqrt{36 - h^2} = \sqrt{25 - h^2}$. Squaring both sides yields $25 - 10\sqrt{36 - h^2} + 36 - h^2 = 25 - h^2$, so that $\sqrt{36 - h^2} = 18/5$ and thus $h = 24/5$. The area of the trapezoid is then $\frac{1}{2} \cdot (10 + 15) \cdot \frac{24}{5} = \boxed{60}$.

4. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Determine the number of real numbers x with $1 < x < 2019$ such that $\lfloor x^3 \rfloor = x^2 \lfloor x \rfloor$.

Answer: 4035.

Solution: Let $\lfloor x \rfloor = n$, so that $n \leq x < n + 1$: then $\lfloor x^3 \rfloor = n^3 + k$ for some integer k with $0 \leq k \leq 3n^2 + 3n$.

The given information implies $x^2 = \frac{\lfloor x^3 \rfloor}{\lfloor x \rfloor} = \frac{n^3 + k}{n} = n^2 + \frac{k}{n}$, meaning that $x = n\sqrt{1 + \frac{k}{n^3}}$. In particular, since $\lfloor x \rfloor = n$, we must have $x^2 \leq n^2 + 2n$ and thus $0 \leq k \leq 2n^2$.

Then we can write $x^3 = \left(n^2 + \frac{k}{n}\right) \cdot n\sqrt{1 + \frac{k}{n^3}} = (n^3 + k) \cdot \sqrt{1 + \frac{k}{n^3}}$, and so $\lfloor x^3 \rfloor = n^3 + k$ will hold if and only if $\left(\sqrt{1 + \frac{k}{n^3}} - 1\right) \cdot (n^3 + k) < 1$, or equivalently, if $\frac{k}{n^3} \cdot (n^3 + k) \leq \sqrt{1 + \frac{k}{n^3}} + 1$, which is the same as $k + \frac{k^2}{n^3} \leq \sqrt{1 + \frac{k}{n^3}} + 1$.

Then we have $k + \frac{k^2}{n^3} \leq \sqrt{1 + \frac{k}{n^3}} + 1 \leq 2 + \frac{k}{2n^3} < 2 + \frac{1}{n}$, so in particular $k < 3$. But notice that $k = 2$ also does not work, and so we have only the possibilities $k = 0$ and $k = 1$. Clearly $k = 0$ will always work, and $k = 1$ will also always work, because $1 + \frac{1}{n^3} \leq 2 < \sqrt{1 + \frac{1}{n^3}} + 1$.

Therefore, the real numbers $x > 1$ such that $[x^3] = x^2[x]$ are precisely the numbers of the form $x = n$ or $x = \sqrt{n^2 + \frac{1}{n}}$ for an integer $n \geq 1$. Excluding the value $x = 1$, this yields 2017 values of the first type and 2018 values of the second type, for a total of 4035.

5. Ken and Tony play a game involving a pile of N candies. On each turn, a player may take either one-third or one-fourth of the remaining candies, rounded up to the nearest integer: thus, if there were 26 candies, a player could either remove 9 or 7 of them. Ken and Tony alternate turns, with Ken going first. In ruleset K, the player who takes the last candy wins, while in ruleset T, the player who takes the last candy loses. Starting with a pile of N candies, prove that Ken has a winning strategy in ruleset K while Tony has a winning strategy in ruleset T if and only if $N = 2^n - 1$ for some positive integer n .

Motivation: We make a table of some small values of N and label which moves are available if there are N candies in the pile. We can then recursively identify each position (in each ruleset) as winning or losing: a position is winning if there is an available move to a losing position, and a position is losing if all available moves are to winning positions.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Can take	1	1	1	1,2	2,2	2,2	2,3	2,3	3,3	3,4	3,4	3,4	4,5	4,5	4,5	4,6	5,6	5,6
Move to	0	1	2	3,2	3	4	5,4	6,5	6	7,6	8,7	9,8	9,8	10,9	11,10	12,10	12,11	13,12
Win K?	W	L	W	W	L	L	W	W	W	W	L	L	L	L	W	W	W	W
Win T?	L	W	L	W	W	L	L	W	W	W	W	L	L	L	L	W	W	W
N	19	20	21	22	23	24	25	26	27	28	29	30	31	32				
Can take	5,7	5,7	6,7	6,8	6,8	6,8	7,9	7,9	7,9	7,10	8,10	8,10	8,11	8,11				
Move to	14,12	15,13	15,14	16,14	17,15	18,16	18,16	19,17	20,18	21,18	21,19	22,20	23,20	24,21				
Win K?	W	W	W	W	L	L	L	L	L	L	L	L	W	W				
Win T?	W	W	W	W	W	W	L	L	L	L	L	L	L	L				

In both rulesets, the winning and losing positions appear in streaks of increasing length. In ruleset T, the streaks of wins start at powers of 2 (2, 4, 8, 16, 32) and double in length each time, while in ruleset K they appear one position earlier. This suggests that the winning positions in ruleset K are those with $2^n - 1 \leq N \leq 2^n + 2^{n-1} - 2$ for some positive integer n , and the winning positions in ruleset T are those with $2^n \leq N \leq 2^n + 2^{n-1} - 1$ for some positive integer n .

Solution: We will prove that the winning positions in ruleset K are those with $2^n - 1 \leq N \leq 2^n + 2^{n-1} - 2$ for some positive integer n , and the winning positions in ruleset T are those with $2^n \leq N \leq 2^n + 2^{n-1} - 1$ for some positive integer n by induction on n . For the base cases $n \leq 4$ we refer to the calculations above. For the inductive step in ruleset K, we must show that if $2^n - 1 \leq N < 2^n + 2^{n-1} - 1$ then N is a winning position and if $2^n + 2^{n-1} - 1 \leq N < 2^{n+1} - 1$ then N is a losing position.

- First, if $2^n - 1 \leq N < 2^n + 2^{n-1} - 1$ then $2^n - 2^{n-2} - 1 \leq N - \lceil N/4 \rceil \leq 2^n - 2^{n-4} - 1$ and so taking $\lceil N/4 \rceil$ stones is a winning move, since all positions in that range are losing positions.
- Second, if $2^n + 2^{n-2} - 1 \leq N < 2^n + 2^{n-1} - 1$ then $2^n - 2^{n-1}/3 - 2 \leq N - \lceil N/3 \rceil < 2^n - 1$ and so taking $\lceil N/3 \rceil$ stones is a winning move, since all positions in that range are losing positions.
- Finally, if $2^n + 2^{n-1} - 1 \leq N < 2^{n+1} - 1$ then both moves are losing moves, because $2^n - 1 \leq N - \lceil N/3 \rceil \leq N - \lceil N/4 \rceil \leq 2^n + 2^{n-1} - 1$ and all positions in this range are winning positions.

For the inductive step in ruleset T, we must show that if $2^n \leq N < 2^n + 2^{n-1}$ then N is a winning position and if $2^n + 2^{n-1} \leq N < 2^{n+1}$ then N is a losing position.

- First, if $2^n \leq N \leq 2^n + 2^{n-2}$ then $2^n - 2^{n-2} \leq N - \lceil N/4 \rceil \leq 2^n - 2^{n-4}$ and so taking $\lceil N/4 \rceil$ stones is a winning move, since all positions in that range are losing positions.
- Second, if $2^n + 2^{n-2} \leq N < 2^n + 2^{n-1}$ then $2^n - 2^{n-1}/3 - 1 \leq N - \lceil N/3 \rceil < 2^n$ and so taking $\lceil N/3 \rceil$ stones is a winning move, since all positions in that range are losing positions.
- Finally, if $2^n + 2^{n-1} \leq N < 2^{n+1}$ then both moves are losing moves, because $2^n \leq N - \lceil N/3 \rceil \leq N - \lceil N/4 \rceil \leq 2^n + 2^{n-1}$ and all positions in this range are winning positions.

Taken together, these results indicate that Ken has a winning strategy in ruleset K while Tony has a winning strategy in ruleset T precisely when $N = 2^n - 1$ for some positive integer n , because these are the only integers satisfying $2^n - 1 \leq N \leq 2^n + 2^{n-1} - 2$ but not $2^n \leq N \leq 2^n + 2^{n-1} - 1$.

Remark: There are various other observations that can simplify the analysis of this game: for example, an alternate formulation of the candy-removal procedure is that if there are currently N candies, then the possible moves are to $f(N) = \lfloor 2N/3 \rfloor$ or to $g(N) = \lfloor 3N/4 \rfloor$. Also, one may show that $f(g(N)) = g(f(N)) = \lfloor N/2 \rfloor$ whenever N is odd, which can allow for a variant formulation of the induction argument based around alternating the two types of moves once a winning or losing position of the appropriate type is reached.

Note: The authors would like to thank Jordan Ellenberg for initially suggesting the idea of this game to them, and also for observing that the winning positions in ruleset T can be characterized as those integers whose base-2 representation starts with “10”.

6. What is the smallest positive integer d such that there exist distinct lattice points A , N , and G on the circle $x^2 + y^2 = d$ with $\cot \angle ANG = 2019$?

Answer: $2038181 = 1009^2 + 1010^2$.

Solution: By the extended law of sines in $\triangle ANG$, we have $\frac{AG}{\sin \angle ANG} = 2\sqrt{d}$. Squaring both sides yields $AG^2 \cdot \csc^2 \angle ANG = 4d$, and using the Pythagorean identity $\csc^2 x = 1 + \cot^2 x$ then produces $4d = AG^2 \cdot (1 + \cot^2 \angle ANG) = AG^2 \cdot (1 + 2019^2)$.

Since A and G are lattice points, the smallest possible value for AG^2 is 2: we cannot have $AG^2 = 1$ since then if $A = (a, b)$ then without loss of generality we would have $G = (a+1, b)$, but these two points cannot lie on the same circle centered at the origin unless $a = -1/2$.

If $AG^2 = 2$ and $A = (a, b)$, without loss of generality we may assume $B = (a+1, b+1)$. Then having both points lie on $x^2 + y^2 = d$ requires $a^2 + b^2 = (a+1)^2 + (b+1)^2$ so that $b = -a-1$, and then $d = a^2 + (a+1)^2$.

From the relation above we require $d = \frac{AG^2}{4}(1 + 2019)^2 = 2038181$, and in fact $2038181 = 1009^2 + 1010^2$ is the sum of two consecutive squares, so for example there exist lattice points $A = (1009, 1010)$ and $G = (1010, 1009)$ on the circle with $AG^2 = 2$. We can then choose N to be any other lattice point on the circle, such as $(1009, -1010)$ or $(230, 1409)$.

Therefore, since any other such circle would necessarily have a larger value of AG^2 , we conclude that $d = \boxed{2038181}$ is the smallest possible value.

Remark: Because $\cot(\angle ANG)$ is a rational function of the coordinates of A , N , and G , there is always a value of d and three lattice points on $x^2 + y^2 = d$ such that $\cot(\angle ANG)$ takes any particular rational value. However, in general it appears to be very difficult to compute the minimal such d , except in special circumstances such as those in this particular problem. (For a challenge, try finding a value of d and three lattice points A, N, G on the corresponding circle with $\cot \angle ANG = 2018$.)