Vermont Mathematics Talent Search, Solutions to Test 4, 2018-2019

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1. Find the smallest positive integer n such that for every integer m with $0 < m < 2019$ there exists an integer k for which $\frac{m}{2019} < \frac{k}{n} < \frac{m+1}{2020}$.

   Answer: $n = 4039$.

   Solution: First observe that the condition with $m = 2018$ yields $\frac{2018}{2019} < \frac{k}{n} < \frac{2019}{2020}$.

   For this to hold we must have $0 < 2019k - 2018n$ and also $0 < 2019n - 2020k$. Since both of these quantities are integers we then require $1 \le 2019k - 2018n$ and also $1 \le 2019n - 2020k$.

   Multiplying the first inequality by 2020, the second by 2019, and adding yields

   $$4039 \le 2020 + 2019 \le 2020(2019k - 2018n) + 2019(2019n - 2020k) = (2019^2 - 2020 \cdot 2018)n = n.$$

   Therefore, we must have $n \ge 4039$.

   On the other hand, it is not hard to see that if $n = 4039$, then $\frac{m}{2019} < \frac{2m+1}{4039} < \frac{m+1}{2020}$ for any $0 < m < 2019$. This can be verified by clearing denominators, or more intuitively by noticing that the middle term is the overall success rate obtained by combining m successes in 2019 attempts with $m+1$ successes in 2020 attempts, and is therefore always between them.

   Hence $n = 4039$ works, so the smallest possible value of n is $\boxed{4039}$.

2. Each of the 8 faces of a hexagonal prism is randomly painted green, gold, black, or white. Find the probability that the prism has at least one pair of green faces that share a side.

   Answer: 33703/65536.

   Solution: We count the probability that the desired event fails to occur based on the number of green faces that are painted:

   - If there are 0 green faces then there is no green pair. Each of the 8 faces can be any of the 3 remaining colors, so there are 3^8 possibilities in this case.
   - If there is 1 green face, then there is no green pair. There are 8 choices for the green face and each of the other 7 can be any of the 3 remaining colors, so there are $8 \cdot 3^7$ possibilities in this case.
   - If there are 2 green faces, then there are $\binom{8}{2} = 28$ choices for the green faces. Of these, 18 result in an adjacent pair of green faces: we could either paint the bottom or top green along with any of the 6 lateral faces (12 ways in total), or we could paint two adjacent lateral faces green (6 ways). Thus there are 10 choices that do not result in an adjacent pair. The remaining 6 faces can be any of the 3 other colors, so there are $10 \cdot 3^6$ possibilities that result in having no adjacent pair of green faces.
   - If there are 3 green faces, if we paint the bottom or top green, then at least one lateral face is also green so we have an adjacent pair. If only lateral faces are painted, then we will have an adjacent pair unless we paint three alternating lateral faces green, and there are 2 ways to do this. The remaining 5 faces can be any of the 3 other colors, so there are $2 \cdot 3^5$ possibilities in this case.

   In total, there are 4^8 possible ways to paint the faces, and of these $3^8 + 8 \cdot 3^7 + 10 \cdot 3^6 + 2 \cdot 3^5 = 131 \cdot 3^5$ result in not having an adjacent pair of green faces. Thus, the desired probability is $1 - \frac{131 \cdot 3^5}{4^8} = \boxed{\frac{33703}{65536}}$.

3. In triangle ABC, $AB = 13$, $AC = 15$, and $BC = 14$. If the bisector of angle B intersects the inscribed circle of $\triangle ABC$ at points D and E, find the length of segment DE.

Answer: 8.

Solution 1: Observe that the center of the inscribed circle of $\triangle ABC$ is the intersection of the angle bisectors, and so in particular the bisector of angle B passes through the center of the circle. Therefore, D and E are two endpoints of a diameter of the inscribed circle. Since the semiperimeter of $\triangle ABC$ is $s = \frac{1}{2}(13+14+15) = 21$, by Heron's formula the area of $\triangle ABC$ is $K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. By the inradius formula, we then have $r = \frac{K}{s} = 4$, and so the length of DE is $2r = \boxed{8}$.

Solution 2: We use coordinates. By the area calculation in Solution 1, we see that the altitude to side BC has length $\frac{2 \cdot 84}{14} = 12$, and by the Pythagorean theorem it divides side BC into segments of length 5 and 9. Thus, we may place vertex B at $(0,0)$, vertex C at $(14,0)$, and vertex A at $(5,12)$, and then side AB has equation $y = \frac{12}{5}x$ and side AC has equation $y = -\frac{4}{3}(x-14)$.

By the angle bisector theorem, the bisector of angle B divides side AC in the ratio $AB : BC = 13 : 14$, so it intersects side AC at the point $\frac{14}{27}(5,12) + \frac{13}{27}(14,0) = (\frac{28}{3}, \frac{56}{9})$, and therefore has equation $y = \frac{2}{3}x$. Similarly, the bisector of angle A divides side BC in the ratio $AB : AC = 13 : 15$, so it intersects side BC at the point $\frac{15}{28}(0,0) + \frac{13}{28}(14,0) = (\frac{13}{2}, 0)$, and therefore has equation $y = -8(x - \frac{13}{2})$.

These two angle bisectors intersect at the center of the inscribed circle, and solving for the intersection point shows that it is $(x,y) = (6,4)$. Then since the inradius is 4, the inscribed circle has equation $(x-6)^2 + (y-4)^2 = 4^2$. The bisector of angle B has equation $y = \frac{2}{3}x$, so plugging into the equation of the circle yields $(x-6)^2 + (\frac{2}{3}x - 4)^2 = 4^2$ so that $\frac{13}{9}x^2 - \frac{52}{3}x + 36 = 0$, and so $x = 6 \pm \frac{12\sqrt{13}}{13}$ after simplifying.

Thus, the two points of intersection are $\left(6 + \frac{12\sqrt{13}}{13}, 4 + \frac{8\sqrt{13}}{13}\right)$ and $\left(6 - \frac{12\sqrt{13}}{13}, 4 - \frac{8\sqrt{13}}{13}\right)$. By the distance formula, the desired distance is then $\sqrt{\left(2 \cdot \frac{12\sqrt{13}}{13}\right)^2 + \left(2 \cdot \frac{8\sqrt{13}}{13}\right)^2} = \sqrt{64} = \boxed{8}$.

4. If $p(x)$ is a polynomial with integer coefficients and n is an integer such that $p(n) = 3$, $p(n+3) = 24$, and $p(2n+3) = 63$, find all possible values of n.

Answer: $n = -13, -1, 1, 3$.

Solution: First observe that if $p(x)$ is a polynomial with integer coefficients and a and b are integers, then $a - b$ must divide $p(a) - p(b)$. Applying this result for $(a,b) = (n+3, n)$, $(2n+3, n)$, and $(2n+3, n+3)$ yields, respectively, that 3 must divide 21, that $n+3$ must divide 60, and that n must divide 39. The latter condition implies that n must be in the set $\{-39, -13, -3, -1, 1, 3, 13, 39\}$, and of these numbers, $\{-13, -1, 1, 3\}$ have the property that $n+3$ divides 60.

For each of these choices, there does exist a quadratic polynomial with integer coefficients passing through the three points $(n, 3)$, $(n+3, 24)$, and $(2n+3, 63)$:

n	-13	-1	1	3
$p(t)$	$(t+15)^2 - 1$	$-23t^2 + 30t + 56$	$8t^2 - 33t + 28$	$t^2 - 2t$

Thus, the possibilities are $n = \boxed{-13, -1, 1, 3}$.

5. The value of $\cos^2(1^\circ)+\cos^2(2^\circ)+\cdots+\cos^2(44^\circ)$ can be written in the form $\dfrac{p+\cot(q^\circ)}{r}$, where p, q, and r are positive integers and $q<90$. Find the ordered triple (p,q,r).

Answer: $(p,q,r)=(87,1,4)$.

Solution 1: Since $\cos^2(x)=\dfrac{\cos(2x)+1}{2}$, we may equivalently evaluate the sum $S'=\cos(2^\circ)+\cos(4^\circ)+\cdots+\cos(88^\circ)$; the original sum S is then equal to $\dfrac{1}{2}S'+22$. Multiplying S' by $\sin(1^\circ)$ yields

$$\begin{aligned} S'\cdot\sin(1^\circ) &= \cos(2^\circ)\sin(1^\circ)+\cos(4^\circ)\sin(1^\circ)+\cdots+\cos(88^\circ)\sin(1^\circ) \\ &= \frac{1}{2}[\sin(3^\circ)-\sin(1^\circ)]+\frac{1}{2}[\sin(5^\circ)-\sin(3^\circ)]+\cdots+\frac{1}{2}[\sin(89^\circ)-\sin(87^\circ)] \\ &= \frac{1}{2}\sin(89^\circ)-\frac{1}{2}\sin(1^\circ) \end{aligned}$$

where we repeatedly used the product-to-sum identity $\cos(a)\sin(b)=\dfrac{1}{2}[\sin(a+b)-\sin(a-b)]$. Since $\sin(89^\circ)=\cos(1^\circ)$, dividing back through by $\sin(1^\circ)$ yields $S'=\dfrac{1}{2}\cot(1^\circ)-\dfrac{1}{2}$, so that $S=\dfrac{87+\cot(1^\circ)}{4}$ and $(p,q,r)=\boxed{(87,1,4)}$.

Solution 2: As in Solution 1, we will instead evaluate the sum $S'=\cos(2^\circ)+\cos(4^\circ)+\cdots+\cos(88^\circ)$. Observe that $\cos(n^\circ)$ is the real part of $e^{2\pi i n/360}=\omega^n$, where $\omega=e^{2\pi i/360}$. Thus,

$$\begin{aligned} S' &= \mathrm{Re}(\omega^2+\omega^4+\cdots+\omega^{88}) \\ &= \mathrm{Re}(1+\omega^2+\cdots+\omega^{88})-1 \\ &= \mathrm{Re}\left(\frac{1-\omega^{90}}{1-\omega^2}\right)-1 \\ &= \mathrm{Re}\left(\frac{1-i}{1-\omega^2}\cdot\frac{\omega^{-1}}{\omega^{-1}}\right)-1 \\ &= \mathrm{Re}\left(\frac{(1-i)(\cos(1^\circ)-i\,\sin(1^\circ))}{-2i\,\sin(1^\circ)}\right)-1 \\ &= \mathrm{Re}\left(\frac{\cos(1^\circ)+\sin(1^\circ)}{2\sin(1^\circ)}+\frac{\cos(1^\circ)-\sin(1^\circ)}{2\sin(1^\circ)}i\right)-1 \\ &= \frac{1}{2}\cot(1^\circ)-\frac{1}{2} \end{aligned}$$

from which we see $S=\dfrac{87+\cot(1^\circ)}{4}$ so that $(p,q,r)=\boxed{(87,1,4)}$ as in Solution 1.

6. Determine, with proof, all positive integers n with the property that there exists a way to label every lattice point in the plane with a positive integer such that

   (a) Every positive integer is used to label exactly one point, and

   (b) The sum of the labels of the vertices of any square is divisible by n.

Answer: $n = 1, 2, 4$.

Solution: Let e be any label and consider the 3×3 square centered at e:

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$$

Then each of $a+b+d+e$, $b+c+e+f$, $d+e+g+h$, $e+f+h+i$ are divisible by n, as are $a+c+g+i$ and $b+d+f+h$. Therefore, $(a+b+d+e)+(b+c+e+f)+(d+e+g+h)+(e+f+h+i)-(a+c+g+i)-(b+d+f+h) = 4e$ must be divisible by n. If n does not divide 4, then this implies every label is divisible by $n/\gcd(4,n) > 1$, which is impossible, because every positive integer must be used as a label somewhere. The only remaining cases are those with $n|4$, so that $n = 1$, 2, or 4. It is clearly sufficient to describe a construction when $n = 4$, since the same example will also work for $n = 1$ and $n = 2$.

It is enough to show that we can create such labelings using residue classes modulo 4 provided each residue class is used infinitely many times: then we can simply replace each residue class with a unique positive integer lying in that residue class. It is not hard to see that there is at least one way to convert these mod-4 labelings to integer labelings: one way is to draw a spiral outward from $(0,0)$ that passes through each lattice point, and then label the kth element on the spiral that lies in the residue class a mod 4 with $a + 4(k-1)$.

Now we claim that labeling the point (a,b) with $a+b$ mod 4 will have the desired property. To see this, suppose one vertex of the square is (x,y) and the next vertex counterclockwise from it is $(x+c, y+d)$. Then the vertex clockwise from (x,y) must be a distance $\sqrt{c^2+d^2}$ from it along the line with slope $-c/d$, and so it is the point $(x-d, y+c)$. The last vertex is then $(x+c-d, y+c+d)$, so the four vertices are (x,y), $(x+c, y+d)$, $(x-d, y+c)$, $(x+c-d, y+c+d)$.

Alternatively, one could use vectors to compute the coordinates of the square's vertices: if one vertex is (x,y) and the next vertex counterclockwise is $(x,y)+(c,d)$, then the vector representing the side of the square is (c,d) and its rotation $\pi/2$ radians counterclockwise is $(-d,c)$, since it is easy to see that this vector is orthogonal to (c,d) and has the correct orientation. Thus the vertex opposite (x,y) is $(x,y)+(c,d)+(-d,c) = (x+c-d, y+d-c)$, and then the remaining vertex is $(x,y)+(-d,c) = (x-d, y+c)$ as claimed.

In any case, the sum of the four labels is $(x+y)+(x+c+y+d)+(x-d+y+c)+(x+c-d+y+c+d) = 4x+4y+4c$, which is indeed divisible by 4. This mod-4 labeling has the required property, so we are done. For illustration, we include a portion of one possible arrangement with $n = 4$:

$$\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 65 & 62 & 63 & 64 & 61 & 58 & 59 & 60 & 57 & \\
 & 68 & 37 & 38 & 35 & 32 & 33 & 34 & 31 & 56 & \\
 & 67 & 36 & 17 & 14 & 15 & 16 & 13 & 30 & 55 & \\
 & 66 & 39 & 20 & 5 & 6 & 3 & 12 & 29 & 54 & \\
\cdots & 69 & 42 & 19 & 4 & 1 & 2 & 11 & 28 & 53 & \cdots \\
 & 72 & 41 & 18 & 7 & 8 & 9 & 10 & 27 & 52 & \\
 & 71 & 40 & 21 & 22 & 23 & 24 & 25 & 26 & 51 & \\
 & 70 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 & \\
 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 & 81 & \\
 & & & & & \vdots & & & & &
\end{array}$$

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