Vermont Mathematics Talent Search, Solutions to Test 1, 2019-2020

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1. Erika has a special calculator that can do any of the basic arithmetic operations (addition, subtraction, negation, multiplication, division, exponentiation, and square roots), but only has four digit buttons 1, 3, 4, and 6. Operations may be used multiple times, but each digit button must be pressed exactly once during a calculation. Four example calculations Erika could make are given below.

Input	Result
$4 \times 6 + 3 + 1$	28
$6 - (4 \times 3 + 1)$	-7
$\sqrt{4^6} - 3 \div 1$	63
$(4+1)^{6-3}$	125

Using her calculator, how many of the positive integers from 1 to 20 inclusive can Erika create?

Answer: 20.

Solution: In fact, Erika can make every one of the positive integers from 1 to 20 inclusive! Here are some possibilities yielding each of these values:

Value	Input 1	Input 2	Input 3
1	$3+4-6\times 1$	$3\times 4 \div 6 - 1$	1^{3+4+6}
2	$3+4-6+1$	$1 \times 3 \times 4 \div 6$	$\sqrt{4\times(6-3+1)}$
3	$3\times(6-4-1)$	$4-6\div 3+1$	3^{6-4-1}
4	$6+3-4-1$	$4\times(6\div 3-1)$	$4^{6\div 3-1}$
5	$6+3-4\times 1$	$\sqrt{\sqrt{6^4}}-1^3$	$6-1^{3+4}$
6	$6+3-4+1$	$4+(6\times 1)\div 3$	$6\times(3+1)\div 4$
7	$6+4-3\times 1$	$4+6\div 3+1$	$6+1^{3+4}$
8	$6+3-4-1$	$(6\div 3)\times 4\times 1$	$\sqrt{\sqrt{4^6}}\times 1^3$
9	$(6\div 3)\times 4+1$	$6+4-1^3$	$3^{6-4\times 1}$
10	$(6-3)\times(4+1)$	$6+4\times 1^3$	$\sqrt{\sqrt{4^6}}+3-1$
11	$3\times(6-1)-4$	$\sqrt{\sqrt{4^6}}+3\times 1$	$3\times 4-1^6$
12	$6+3+4-1$	$\sqrt{6^4}\div 3\times 1$	$3\times 4\times 1^6$
13	$6\times 3-4-1$	$\sqrt{6^4}\div 3+1$	$3\times 4+1^6$
14	$6+3+4+1$	$6\times 3-4\times 1$	$6\times\sqrt{4}+3-1$
15	$6\times 3-4+1$	$6\times 3-\sqrt{4}-1$	$6\times\sqrt{4}+3\times 1$
16	$4\times(6-3+1)$	$6\times 3-\sqrt{4}\times 1$	$\sqrt{4}\times(6+3-1)$
17	$4\times(6-1)-3$	$6\times 3-\sqrt{4}+1$	$6\times 3-1^4$
18	$\sqrt{6^4}\div(3-1)$	$6\times\sqrt{3\times(4-1)}$	$6\times 3\times 1^4$
19	$3\times(6-1)+4$	$6\times 3+\sqrt{4}-1$	$6\times 3+1^4$
20	$6\times 4-3-1$	$6\times 3+\sqrt{4}\times 1$	$(6-1^3)\times 4$

Since Erika can make each of these integers, the answer is $\boxed{20}$.

Remark: In fact, each of the integers from 1 to 40 inclusive can be obtained with Erika's calculator, along with quite a few others. (Try to see how many you can make!)

2. Under a court order, a certain company was mandated to produce 17 million pages of internal documents. Given that a ream of paper contains 500 pages, that a box of 10 reams of paper measures 12 inches by 18 inches by 10 inches, and that a standard shipping container measures 8 feet by 8.5 feet by 20 feet, what is the smallest number of whole standard shipping containers that would be needed to contain all of the documents?

Answer: 4.

Solution: First, we see that 17 million pages corresponds to $\frac{17000000}{500 \cdot 10} = 3400$ boxes of paper. We also calculate that a standard shipping container's volume is equal to $8 \cdot 8.5 \cdot 20 = 1360$ cubic feet, and one box of paper takes up $1 \cdot 1.5 \cdot \frac{5}{6} = 1.25$ cubic feet. Thus, one shipping container can hold $\frac{1360}{1.25} = 1088$ boxes of paper in total. Since $\frac{3400}{1088} = 3.125$ is between 3 and 4, this means that $\boxed{4}$ standard shipping containers would be required to hold all of the documents.

3. In the magic square below, all of the entries are prime numbers forming an arithmetic progression, and the largest value has been given. Fill in the remaining entries, given that the upper-right entry is the second-largest prime.

2089		

Answer:

1039	829	1879
2089	1249	409
619	1669	1459

Solution: From the information given, we see that the nine entries in the magic square are n, $n+a$, $n+2a$, ... , $n+8a = 2089$ for some positive integers n and a. Note that n cannot equal 2, 3, 5, or 7, since $2089 - n$ would not be a multiple of 8 in any of those cases, so we must have $n \geq 11$.

If a is odd, then at least one of $n, n+a$ is even, and thus since $n > 2$ one of them would be non-prime: thus, a is divisible by 2. Similarly, if a is not a multiple of 3, then at least one of $n, n+a, n+2a$ is a multiple of 3, and again since $n > 3$ one of them would be non-prime: thus, a is divisible by 3. In the same way, we see that a must be divisible by 5 (otherwise one of $n, n+a, n+2a, n+3a, n+4a$ would be a multiple of 5) and also by 7 (otherwise one of $n, n+a, \ldots, n+6a$ would be a multiple of 7).

Therefore, a must be divisible by $2 \cdot 3 \cdot 5 \cdot 7 = 210$. But since we must have $8a < n + 8a = 2089$ so that $a < 2089/8 = 261.125$, the only possibility is to have $a = 210$ with $2089 - 8 \cdot 210 = 409$: then the nine numbers in the magic square must be 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089.

It is a standard fact that (up to rotation and reflection) there is a unique magic square formed using the digits 1-9, namely

4	3	8
9	5	1
2	7	6

and therefore by translating and rescaling we see that there is a unique magic square formed using any 9 numbers in arithmetic progression. By performing the appropriate translation and rescaling, we obtain the desired magic square (which is unique because the largest entry is in the left column and the second-largest entry is in the upper-right corner):

1039	829	1879
2089	1249	409
619	1669	1459

Remark: It is tedious, but not difficult, to verify that all 9 entries are actually prime. (In fact, 199 is also prime, so subtracting 210 from each entry also yields a magic square of prime numbers.)

4. Two sides of a triangle have lengths 12 and 15, and the area of the triangle is $10\sqrt{65}$. Find all possible values for the length of the remaining side of the triangle.

Answer: $\sqrt{209}$, 23.

Solution 1: Suppose θ is the angle between the sides of lengths 12 and 15: then the area of the triangle is $10\sqrt{65} = \frac{1}{2} \cdot 12 \cdot 15 \sin(\theta)$, so that $\sin(\theta) = \frac{\sqrt{65}}{9}$. Then $\cos^2(\theta) = 1 - \sin^2(\theta) = \frac{16}{81}$ so that $\cos(\theta) = \pm\frac{4}{9}$. Then by the law of cosines, the remaining side length c of the triangle satisfies $c^2 = 12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cdot \cos(\theta)$, so we obtain $c^2 = 12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cdot \frac{4}{9} = 209$ so that $c = \sqrt{209}$, or $c^2 = 12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cdot (-\frac{4}{9}) = 529$ so that $c = \sqrt{529} = 23$. Thus, the side lengths are $c = \boxed{\sqrt{209} \text{ or } 23}$.

Solution 2: If the remaining side has length c, the semiperimeter is $(27+c)/2$. Then Heron's formula for the area yields $10\sqrt{65} = \sqrt{\frac{27+c}{2} \cdot \frac{27-c}{2} \cdot \frac{3+c}{2} \cdot \frac{-3+c}{2}}$. Squaring both sides and collecting terms yields $16 \cdot 100 \cdot 65 = (27^2 - c^2)(c^2 - 3^2)$. Setting $d = c^2$ and rearranging yields a quadratic $-81^2 + 738d - d^2 = 104000$ so that $d^2 - 738d + 110561 = 0$. This yields $d = 369 \pm \sqrt{369^2 - 110561} = 369 \pm \sqrt{136161 - 110561} = 369 \pm \sqrt{25600} = 369 \pm 160 = 209, 529$. Thus, the side lengths are $c = \boxed{\sqrt{209} \text{ or } 23}$.

5. A robot is placed on a portion of a grid, pictured below, and programmed to move one unit each minute randomly in one of the four possible directions parallel to the grid (i.e., up, down, left, or right). It starts at the square marked "S", and if it moves into one of the squares marked "x", the robot falls off the grid. What is the probability that the robot will enter the square marked "E" at least once before it falls off the grid?

x	x	x	x	x
x			E	x
x		x		x
x	S			x
x	x	x	x	x

Answer: 1/97.

Solution: Say that a "win" occurs if the robot enters the E square at some point in the future. We label each square with the probability that a win will occur if the robot is currently located at that square:

-	-	-	-	-
-	b	a	1	-
-	c	-	a	-
-	d	c	b	-
-	-	-	-	-

By symmetry, since the paths are symmetric, the probabilities on each side of the center square are equal. Now from the given information, the robot has a 1/4 probability of moving in each possible direction from each labeled square, so we obtain the following system:

$$\begin{aligned} a &= \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot b \\ b &= \frac{1}{4} \cdot a + \frac{1}{4} \cdot c \\ c &= \frac{1}{4} \cdot b + \frac{1}{4} \cdot d \\ d &= \frac{1}{4} \cdot c + \frac{1}{4} \cdot c \end{aligned}$$

Solving the equations from the bottom up in terms of c yields $d = \frac{1}{2}c$, $b = \frac{7}{2}c$, $a = 13c$. Then the top equation yields $13c = \frac{1}{4} + \frac{7}{8}c$ so that $c = \frac{2}{97}$, and then $d = \frac{1}{2}c = \boxed{\frac{1}{97}}$.

6. Four distinct points are chosen in the xy-plane.

(a) If all four points lie on the parabola $y = x^2$, prove that there is a circle passing through the four points if and only if the sum of their x-coordinates is 0.

(b) If all four points lie in the hyperbola $y = 2/x$, prove that there is a circle passing through the four points if and only if the product of their y-coordinates is 4.

Solution (a): Suppose the circle $(x-h)^2 + (y-k)^2 = r^2$ intersects the parabola $y = x^2$ in four distinct points (a, a^2), (b, b^2), (c, c^2), (d, d^2). Then the x-coordinates of the four points are distinct solutions of the equation $(x-h)^2 + (x^2-k)^2 = r^2$, which is equivalent to $x^4 + (1-2k)x^2 - 2hx + (h^2+k^2-r^2) = 0$. The left-hand side of this relation must factor as $(x-a)(x-b)(x-c)(x-d)$, so upon multiplying out and looking at the coefficient of x^3, we conclude that $a+b+c+d = 0$.

For the other direction, suppose we have four distinct points $A(a, a^2)$, $B(b, b^2)$, $C(c, c^2)$, and $D(d, d^2)$ on the parabola $y = x^2$ such that $a+b+c+d = 0$. Let the circumcircle of triangle ABC (note that A, B, C cannot be collinear since they lie on a parabola) have equation $(x-h)^2 + (y-k)^2 = r^2$: then the solutions to the system $(x-h)^2 + (y-k)^2 = r^2$ and $y = x^2$ are the intersection points of the circle and the parabola. Plugging the second equation into the first yields $x^4 + (1-2k)x^2 - 2hx + (h^2+k^2-r^2) = 0$, as above, and by assumption this degree-4 equation has three solutions $x = a$, $x = b$, and $x = c$. We therefore have a factorization $x^4 + (1-2k)x^2 - 2hx + (h^2+k^2-r^2) = (x-a)(x-b)(x-c)(x-e)$, and by comparing coefficients of x^3 we have $a+b+c+e = 0$, whence $e = d$, and so the other intersection point is D, as required.

Solution (b): Suppose the circle $(x-h)^2 + (y-k)^2 = r^2$ intersects the hyperbola $y = 2/x$ in four distinct points $(2/a, a)$, $(2/b, b)$, $(2/c, c)$, $(2/d, d)$. Then the y-coordinates of the four points are distinct solutions of the equation $(2/y-h)^2 + (y-k)^2 = r^2$, which is equivalent to $y^4 - 2ky^3 + (h^2+k^2-r^2)y^2 - 4hy + 4 = 0$. The left-hand side of this relation must factor as $(y-a)(y-b)(y-c)(y-d)$, so upon multiplying out and comparing the constant terms, we conclude that $abcd = 4$.

For the other direction, suppose we have four distinct points $A(2/a, a)$, $B(2/b, b)$, $C(2/c, c)$, $D(2/d, d)$. with $abcd = 4$. Again after noting that A, B, C are not collinear, let the circumcircle of triangle ABC have equation $(x-h)^2 + (y-k)^2 = r^2$: then the solutions to the system $(x-h)^2 + (y-k)^2 = r^2$ and $y = 1/x$ are the intersection points of the circle and the hyperbola. Plugging the second equation into the first yields $y^4 - 2ky^3 + (h^2+k^2-r^2)y^2 - 4hy + 4 = 0$ as above, and as in Solution 1(a), we have a factorization $y^4 - 2ky^3 + (h^2+k^2-r^2)y^2 - 4hy + 4 = (y-a)(y-b)(y-c)(y-e)$, and by comparing constant terms we have $abce = 4 = abcd$. Since none of a, b, c can be zero, we conclude $e = d$, and so the other intersection point is D, as required.

Remark: There are various other geometric characterizations of cyclic quadrilaterals, such as (i) the perpendicular bisectors of the four sides are concurrent, (ii) the quadrilateral satisfies the conditions of Ptolemy's theorem, namely that $AB \cdot CD + AD \cdot BC = AC \cdot BD$, and (iii) opposite angles sum to π radians. Each of these conditions can be used to give a more geometrically-flavored solution to both parts of the problem, and many approaches of this type were submitted by students.