Vermont Mathematics Talent Search, Solutions to Test 4, 2019-2020

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. Kiran has 2020 red blocks, labeled with the integers from 1 to 2020 inclusive, and Evan has 2020 blue blocks, also labeled with the integers from 1 to 2020 inclusive. Kiran and Evan group their blocks together into pairs, with one red block paired with one blue block, in such a way that the sum of the labels of the blocks in each pair is a power of 2. They then evaluate the product of the numbers on each pair of blocks. What is the sum of the 2020 products that Kiran and Evan obtain?

   Answer: 1 430 125 898.

   Solution: The largest possible sum of two block numbers is $2020 + 2020 = 4040 < 4096 = 2^{12}$. Therefore, the only possible way to pair a block of either color whose label is an integer $n \geq 2^{10}$ is to pair it with the block $2^{11} - n$ to make a sum of $2^{11} = 2048$.

   Thus, we see that the red blocks of labels 2020, 2019, 2018, ... , 1024 must pair with the blue blocks of labels 28, 29, ... , 1024, and likewise the blue blocks of labels 2020, 2019, 2018, ... , 1024 must pair with the red block of labels 28, 29, ... , 1024. Once these pairings are made, the only blocks remaining are red blocks 1-27 and blue blocks 1-27. By the same logic, the only possible pairings of a block with label $n \geq 16$ is with the block $32 - n$, so we pair blocks 5 red-27 blue, 6 red-26 blue, ... , 27 red-5 blue. This leaves the red and blue blocks labeled 1-4, and applying the same logic again shows that the only possible pairing is 4 red-4 blue, 3 red-1 blue, 2 red-2 blue, 1 red-3 blue.

   Then the desired sum is $[2020\cdot 28+2019\cdot 29+\cdots+28\cdot 2020]+[5\cdot 27+6\cdot 26+\cdots+27\cdot 5]+[4\cdot 4]+[3\cdot 1+2\cdot 2+1\cdot 3]$.

   We can evaluate the first two sums using the identity $1\cdot n+2\cdot(n-1)+\cdots+n\cdot 1 = \frac{1}{6}n(n+1)(n+2)$ which can be established either via induction or by using the formulas for $1+2+\cdots+n$ and $1^2+2^2+\cdots+n^2$.

   We get $[2020\cdot 28+2019\cdot 29+\cdots+28\cdot 2020]+[5\cdot 27+6\cdot 26+\cdots+27\cdot 5]+[4\cdot 4]+[3\cdot 1+2\cdot 2+1\cdot 3] = \boxed{1\,430\,125\,898}$.

2. The roots of the polynomial $3x^2 + 7x + k$ are $\sec(\theta)$ and $\tan(\theta)$ for some angle θ with $0 \leq \theta \leq 2\pi$. Determine the value of k.

   Answer: $\frac{580}{147}$.

   Solution: If $\sec(\theta)$ and $\tan(\theta)$ are the roots of $3x^2+7x+k$, then we must have $3x^2+7x+k = 3(x-\sec(\theta))(x-\tan(\theta))$ and so by multiplying out and comparing coefficients we see that $\sec(\theta) + \tan(\theta) = -\frac{7}{3}$.

   Since $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$, we must have $\sec\theta - \tan\theta = -\frac{3}{7}$.

   Adding the equations yields $2\sec(\theta) = -\frac{58}{21}$ and so $\sec(\theta) = -\frac{29}{21}$, and then $\tan(\theta) = -\frac{20}{21}$.

   Finally, we obtain $k = 3\sec(\theta)\tan(\theta) = \boxed{\frac{580}{147}}$.

3. A convex pentagon has consecutive side lengths of d, d, d, d, and $5\sqrt{11}$. If there exists a circle of radius 9 passing through all of its vertices, find all possible values of d^2.

Answer: 27, $162 - 27\sqrt{11}$.

Solution: Write $r = 9$ and $a = 5\sqrt{11}$. Then the angle subtended by each side of length d is $2\arcsin(\frac{d}{2r})$ while the angle subtended by the side of length a is $2\arcsin(\frac{a}{2r})$. If the center of the circle is contained in the pentagon, then the sum of these five angles must be 2π radians, while if the center of the circle is not contained in the pentagon, then one of the angles is equal to the sum of the other four. In the first case, this yields the equation $4\arcsin(\frac{d}{2r}) + \arcsin(\frac{a}{2r}) = \pi$, while in the second case, this yields the equation $4\arcsin(\frac{d}{2r}) = \arcsin(\frac{a}{2r})$.

Collectively, these two cases both imply that $\sin\left[4\arcsin(\frac{d}{2r})\right] = \sin\left[\arcsin(\frac{a}{2r})\right]$. By using the identity $\sin(4\theta) = 4\sin(\theta)\sqrt{1-\sin^2(\theta)}[1 - 2\sin^2(\theta)]$, this equation yields $2 \cdot \frac{d}{2r}\sqrt{1 - \frac{d^2}{4r^2}} \cdot (1 - \frac{d^2}{2r^2}) = \frac{a}{2r}$. Squaring both sides and clearing denominators yields $d^2(4r^2 - d^2) \cdot (2r^2 - d^2)^2 = a^2 r^6$.

Plugging in the given numbers and setting $x = d^2$ then yields the equation $x(324 - x)(162 - x)^2 = 146146275$, so we obtain the equation $x^4 - 648x^3 + 131220x^2 - 8503056x + 146146275 = 0$. Factoring yields $(x - 27)(x - 297)(x^2 - 324x + 18225) = 0$ and so we obtain four possible solutions $x = 27$, 297, and $162 \pm 27\sqrt{11}$.

However, the two solutions $d^2 = 297$ and $d^2 = 162 + 27\sqrt{11}$ are extraneous, because we must have $d < 9\sqrt{2}$ in order for the pentagon to exist. The other two solutions $d^2 = \boxed{27,\ 162 - 27\sqrt{11}}$ do yield actual pentagons.

4. Let d be the greatest common divisor of $2^{2019^{2018}} - 2$ and $2^{2019^{2020}} - 2$. Compute the value of $\log_2(d + 2)$.

Answer: $2019^2 = 4\,076\,361$.

Solution: Clearly, d is even since both $2^{2019^{2018}} - 2$ and $2^{2019^{2020}} - 2$ are even, so write $d = 2k$: then k is the greatest common divisor of $2^{2019^{2018}-1} - 1$ and $2^{2019^{2020}-1} - 1$ and $\log_2(d + 2) = 1 + \log_2(k + 1)$. Next, we apply the following Lemma:

Lemma: If m, a, and b are integers, then $\gcd(m^a - 1, m^b - 1) = m^{\gcd(a,b)} - 1$.

Proof: Write $b = qa + r$ where $0 \le r < a$. Then we can easily check that $m^b - 1 = m^{qa+r} - 1 = m^r(m^{qa} - 1) + (m^r - 1)$, and so since $m^{qa} - 1$ is divisible by $m^a - 1$ because we can write $m^{qa} - 1 = (m^a - 1)(m^{a(q-1)} + m^{a(q-2)} + \cdots + m^a + 1)$, we see that the remainder upon dividing $m^b - 1$ by $m^a - 1$ is $m^r - 1$.

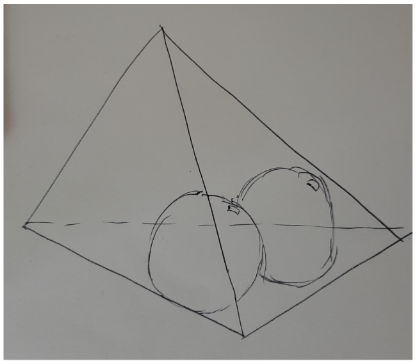
Thus, $\gcd(m^b - 1, m^a - 1) = \gcd(m^a - 1, m^r - 1)$. This means that applying one step of the Euclidean algorithm will not change the gcd of the two powers. Then it is easy to see that applying the full Euclidean algorithm to the two powers will produce a gcd of $m^{\gcd(a,b)} - 1$, as claimed.

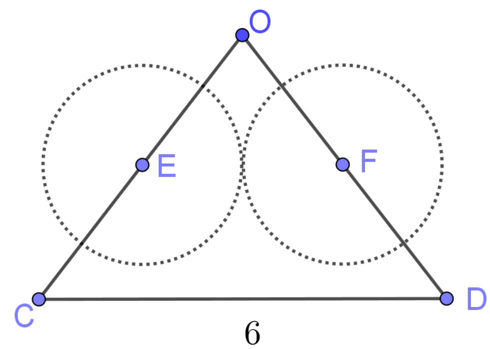
By applying the Lemma with $m = 2$, $a = 2019^{2018}$, and $b = 2019^{2020}$, we see $k = \gcd(2^{2019^{2018}-1} - 1, 2^{2019^{2020}-1} - 1) = 2^{\gcd(2019^{2018}-1, 2019^{2020}-1)} - 1$, so it remains to determine $\gcd(2019^{2018} - 1, 2019^{2020} - 1)$. But by another application of the Lemma with $m = 2019$, $a = 2018$, and $b = 2020$, we see that $\gcd(2019^{2018} - 1, 2019^{2020} - 1) = 2019^{\gcd(2018,2020)} - 1 = 2019^2 - 1$. Therefore, $k = 2^{2019^2-1} - 1$ and so $\log_2(d + 2) = 1 + \log_2(k + 1) = 1 + 2019^2 - 1 = \boxed{2019^2 = 4\,076\,361}$.

5. Two congruent spheres with disjoint interiors are both contained inside a regular tetrahedron of side length 6. Determine the greatest possible value for the shared radius of the spheres.

Answer: $\dfrac{3\sqrt{6}-3}{5}$.

Solution 1: We can see that by sliding the spheres slightly to one side or the other (and then increasing the shared radius if possible) the maximum possible radius will occur when the two spheres are tangent to each other and three of the four faces of the tetrahedron, as shown below.



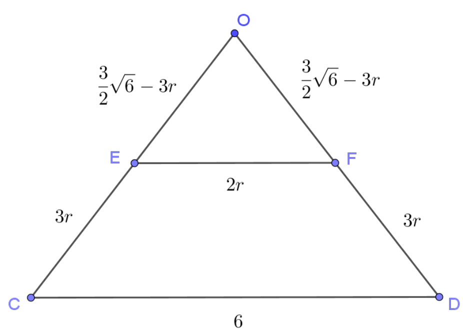


Let the tetrahedron be $ABCD$, and without loss of generality assume that sphere E is tangent to faces A, B, and C and sphere F is tangent to faces A, B, and D (where we label each face by the opposite vertex). Also let the center of the tetrahedron be O and the shared radius of spheres E and F be r. Then by symmetry, E lies in segment OD and F lies on segment OC. A cross-section through plane OCD is given above.

Furthermore, if G is the center of face ACD and K is the point of tangency between sphere E and face ACD, then triangle EKC is similar to triangle OGC.

From standard properties of equilateral triangles and regular tetrahedra (as follows from a few applications of the Pythagorean theorem), we have $OC = OD = \frac{3}{2}\sqrt{6}$, $CG = 2\sqrt{3}$, and $OG = \frac{1}{2}\sqrt{6}$. Therefore, since $\triangle EKC$ is similar to $\triangle OGC$ and $EK = r$, we have $CE = 3r$.

By symmetry we also have $DF = 3r$, and then we obtain $OE = OF = \frac{3}{2}\sqrt{6} - 3r$. Since the spheres are tangent we also have $EF = 2r$. Filling in all these lengths yields the labeled diagram below.



Finally, since $\triangle OEF$ is similar to $\triangle OCD$ by symmetry, we see that $\dfrac{OE}{EF} = \dfrac{OC}{CD}$, which yields the relation $\dfrac{\frac{3}{2}\sqrt{6}-3r}{2r} = \dfrac{\frac{3}{2}\sqrt{6}}{6}$. Clearing the denominators yields $3\sqrt{6} = r(6+\sqrt{6})$, so that $r = \dfrac{3\sqrt{6}}{6+\sqrt{6}} = \dfrac{3}{\sqrt{6}+1} = \boxed{\dfrac{3\sqrt{6}-3}{5}}$ upon rationalizing the denominator.

Solution 2: As above, we first note that maximum possible radius will occur when the two spheres are tangent to each other and three of the four faces of the tetrahedron. We then see that both spheres are separately inscribed in a pyramid obtained by bisecting $ABCD$ along a plane passing through A, B, and the midpoint X of CD. Thus, the radius r is the radius of the sphere inscribed in tetrahedron $ABCX$.

By drawing segments OA, OB, OC, and OX, we divide $ABCX$ into four smaller tetrahedra each with a height r and whose bases are the four respective faces of $ABCX$. Since the sum of the volumes of these tetrahedra is the volume V of $ABCX$, we deduce that $\frac{1}{3}r \cdot ([ABC] + [ABX] + [ACX] + [BCX]) = V$, and therefore $r = \dfrac{3V}{[ABC] + [ABX] + [ACX] + [BCX]} = \dfrac{3 \cdot 9\sqrt{2}}{9\sqrt{3} + 9\sqrt{2} + \frac{9}{2}\sqrt{3} + \frac{9}{2}\sqrt{3}} = \dfrac{3\sqrt{2}}{2\sqrt{3} + \sqrt{2}} =$ $\boxed{\dfrac{3\sqrt{6} - 3}{5}}$ upon rationalizing the denominator.

Remark: The method described in solution 2 yields the formula for the inradius of an arbitrary tetrahedron analogous to the inradius formula $r = \dfrac{2K}{a+b+c}$ for a triangle.

6. Suppose that $\sqrt{5}$ is expressed in base 3, as $2.020101_3 \cdots = 1.d_1d_2d_3d_4d_5d_6 \ldots$. Prove that for every positive integer n, at least one of the base-3 digits d_n, d_{n+1}, d_{n+2}, ... , d_{2n} is nonzero.

Solution: We will show the slightly stronger result that at least one digit d_{n+1}, d_{n+2}, ... , d_{2n} is nonzero. (This was the originally intended version of the problem.)

Thus, suppose otherwise, so that $\sqrt{5}_3 = 2.d_1 \ldots d_{n-1}d_n00 \ldots 0d_{2n+1} \ldots$. If we let k be the base-3 integer $2d_1 \ldots d_n$ then we have $0 \leq \sqrt{5} - \dfrac{k}{3^n} \leq \dfrac{2}{3^{2n+1}} + \dfrac{2}{3^{2n+2}} + \cdots = \dfrac{1}{3^{2n}}$.

Since $0 \leq \dfrac{k}{3^n} \leq \sqrt{5}$, we also see that $0 \leq \sqrt{5} + \dfrac{k}{3^n} \leq 2\sqrt{5}$. Multiplying this inequality with the one from above yields $0 \leq 5 - \dfrac{k^2}{3^{2n}} \leq \dfrac{2\sqrt{5}}{3^{2n}}$, and then clearing the denominators yields $0 \leq 5 \cdot 3^{2n} - k^2 \leq 2\sqrt{5} < 5$.

Since $5 \cdot 3^{2n} - k^2$ is an integer, it must therefore be equal to 0, 1, 2, 3, or 4. It cannot be 0 because $\sqrt{5}$ is irrational.

Furthermore, we can see that $5 \cdot 3^{2n} - k^2 \equiv -k^2$ modulo 5, and therefore it cannot equal 2 or 3 (since the only possible values of $-k^2$ are 0, 1, 4 modulo 5).

Likewise, $5 \cdot 3^{2n} - k^2 \equiv -k^2$ modulo 3, and therefore it cannot equal 1 or 4 (since the only possible values of $-k^2$ are 0, 2 modulo 3).

This eliminates all of the possibilities, and therefore we have reached a contradiction. Thus, at least one of the digits d_{n+1}, d_{n+2}, ... , d_{2n} is nonzero, as claimed.

Remark: This argument can be simplified to prove just the requested result: starting instead with $0 \leq \sqrt{5} - \dfrac{k}{3^{n-1}} \leq \dfrac{1}{3^{2n}}$, we see $0 \leq \sqrt{5} + \dfrac{k}{3^{n-1}} \leq 2\sqrt{5}$, and so $0 \leq 5 - \dfrac{k^2}{3^{2n-2}} \leq \dfrac{2\sqrt{5}}{3^{2n}}$. Then $0 \leq 5 \cdot 3^{2n-2} - k^2 \leq \dfrac{2\sqrt{5}}{9} < 1$, which is an immediate contradiction since $5 \cdot 3^{2n-2} - k^2$ is a nonzero integer.