**Vermont State Mathematics Coalition Talent Search** -- December 2019
Test 3 of the 2019-2020 school year

PRINT NAME:_________________________________________ Signature:_________________
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School________________________________________________ Grade _____________

Current Mathematics Teacher: _______________________________________________

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to Kiran.MacCormick@mvsdschools.org or be postmarked by **January 24, 2020** and submitted to

Kiran MacCormick
Missisquoi Valley Union High School
175 Thunderbird Drive
Swanton, VT 05488

**To receive the next tests via email, clearly print your email address below:**

____________________________________________________________________________

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Missisquoi Valley Union HS) and Evan Dummit (Assistant Teaching Professor at Northeastern University).

**Vermont State Mathematics Coalition Talent Search** -- December 2019
Test 3 of the 2019-2020 school year

1. A set is called "double-free" if it contains no pair of distinct elements $(a, b)$ with $b = 2a$. Find all values of $N$ such that the maximum number of possible elements in a double-free subset of $\{1, 2, 3, ..., N\}$ is 2020.

Answer:___________

2. In triangle $ABC$, $AB = 14$, $AC = 13$, and $BC = 15$, and point $D$ is located on $AB$. If the inscribed circle in $\triangle ACD$ is tangent to the inscribed circle of $\triangle BCD$ at $E$, find the length of $DE$.

Answer:___________

3. Calculate the angle $\alpha$, in radians, with $0 \leq \alpha \leq \pi$, such that

$$\cos(\alpha) = 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{5\pi}{7}\right) - 2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right).$$

Answer:____________

4. A tetrahedron has three edges of length $4\sqrt{3}$ and three edges of length 6. Compute all possible values for the volume of this tetrahedron.

Answer:____________

5. The letters in the word DEPRESSURIZED are randomly permuted. What is the probability that neither the string PRIDE nor the string DEER--with those letters in that order, and with no other letters in between--appears in the resulting permutation?

Answer:___________

6. Suppose $a, b, c, d$ are positive integers such that $a^2 + b + c + d$, $b^2 + a + c + d$, $c^2 + a + b + d$, and $d^2 + a + b + c$ are all perfect squares.

(a) Prove that at least two of $a, b, c, d$ must be equal.

(b) Find all possible ordered quadruples $(a, b, c, d)$.

*Note: For this problem, please include your proof on separate sheets of paper.*

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Missisquoi Valley Union HS) and Evan Dummit (Assistant Teaching Professor at Northeastern University).