Vermont Mathematics Talent Search, Solutions to Test 3, 2020-2021

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. Kiran has 2021 different weights, and he knows that the mass of the nth weight is 2^n grams for $1 \le n \le 2021$. Kiran uses a balance scale that reports the signed difference in the total masses of the weights placed in its two baskets. For example, if in Basket A Kiran places the 1- and 16-gram weights, and in the second basket Kiran places the 32-gram weight, the scale would read $+15$ grams. Kiran places some weights in each basket, and the scale reads $+4082420$ grams, which Kiran notices equals $2020 \cdot 2021$. What is the smallest possible number of weights Kiran could have used in total?

   Answer: 8.

   Solution: Since we are adding powers of 2, we first express $4082420_{10} = 1111100100101011110100_2$ in base 2. If we place the nth object in Basket A, then this corresponds to adding 1 in the 2^n-place. Thus, one possible way of obtaining this number is just to place the objects of appropriate weights where 1 occurs in the base-2 expansion into basket A: this would require a total of 13 weights, one for each 1 in the base-2 expansion of 4082420.

   However, we can do better than this by placing some weights in basket B: if we place the object of mass 2^n grams in Basket B, then in order to have a positive sum, we must counterbalance it with a heavier object in Basket A. If this object has mass 2^{n+k}, then the net weight is $2^n(2^k - 1)$, which corresponds to adding 1 in the nth, $(n+1)$st, ... , $(n+k-1)$st places in the base-2 expansion. This approach uses fewer objects than the method above (one object per 1 in the base-2 expansion) whenever we have a string of 3 or more 1s in the base-2 expansion.

   Thus, we may in fact obtain a signed difference of $+4082420$ grams using a total of $\boxed{8}$ objects: two each for the two runs of 1s, and four for the remaining individual 1s. Explicitly, we have $4082420 = 2^{22} - 2^{17} + 2^{14} + 2^{11} + 2^9 + 2^8 - 2^4 + 2^2$, corresponding to putting the weights 2^{22}, 2^{14}, 2^{11}, 2^9, 2^8, 2^2 in Basket A and 2^{17}, 2^4 in Basket B. It is easy to see that any arrangement of objects in the two baskets can be decomposed as a sum of these two types of differences, and so the minimum possible number of weights is indeed 8.

2. Find all ordered pairs (a, b) of positive integers with such that $\dfrac{2021-a}{a} \cdot \dfrac{2021-b}{b} = 3$.

   Answer: $(a, b) = (94, 1763)$ or $(1763, 94)$.

   Solution: Clearing denominators and rearranging yields $2ab + 2021a + 2021b - 2021^2 = 0$. Multiplying both sides by 2 and factoring then gives $(2a + 2021)(2b + 2021) = 3 \cdot 2021^2$. Note that $3 \cdot 2021^2 = 3 \cdot 43^2 \cdot 47^2$ and $2a + 2021$, $2b + 2021$ are both greater than $2021 = 43 \cdot 47$, and so each of the terms $2a + 2021$ and $2b + 2021$ must have at least two terms from the list $\{43, 43, 47, 47\}$, hence each must have exactly two. Since $(47/43)^2 < 3$, the larger term must have the extra factor of 3. If $a < b$ then since the smaller term must be greater than $43 \cdot 47$, the only possibility is for $2a + 2021 = 47^2$ and then $2b + 2021 = 3 \cdot 43^2$, yielding $(a, b) = (94, 1763)$. Since we also obtain the symmetric pair, we obtain the two solutions $(a, b) = \boxed{(94, 1763),\ (1763, 94)}$.

3. Preston has a deck of 12 special cards: the first card says "Exactly 0 of the statements on the cards to the left of this card are true", the second card says "Exactly 1 of the statements on the cards to the left of this card are true", the third card says "Exactly 2 of the statements on the cards to the left of this card are true", etc., and the twelfth card says "Exactly 11 of the statements on the cards to the left of this card are true". Preston randomly shuffles the 12 cards and then deals them in a line. What is the probability that exactly 4 cards have true statements written on them?

Answer: 1/30.

Solution: Suppose exactly 4 cards have true statements. Reading from the left end of the line of cards, consider the first card with a true statement. None of the cards to its left have true statements written on them, so this card must read "Exactly 0 of the statements on the cards to the left of this card are true". Continuing to the right, in the same way we see the second card with a true statement must say "Exactly 1 ...", the third true card must say "Exactly 2 ...", and the fourth true card must say "Exactly 3 ...". After this card, the card labeled "Exactly 4 ..." cannot appear, because it would provide a fifth card with a true statement on it.

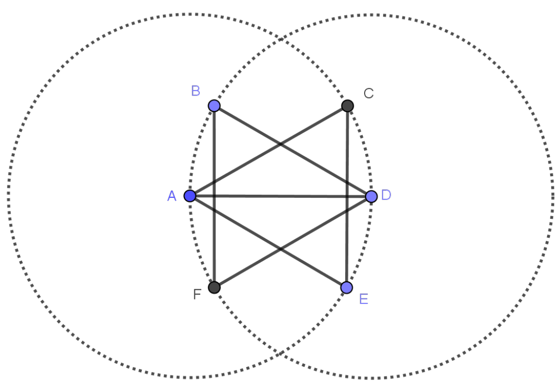
Conversely, if the cards labeled 0-1-2-3 appear in that order, and the card labeled 4 appears to the left of the card labeled 3, then by the logic given above, exactly 4 cards (namely, the cards 0,1,2,3) will have true statements written on them. If we ignore the cards with larger labels, there are 4 possible ways of arranging these five cards from left to right, out of $5! = 120$ total: they are 4-0-1-2-3, 0-4-1-2-3, 0-1-4-2-3, and 0-1-2-4-3. Therefore, the desired probability is $4/5! = \boxed{1/30}$.

4. A convex hexagon has 9 diagonals.

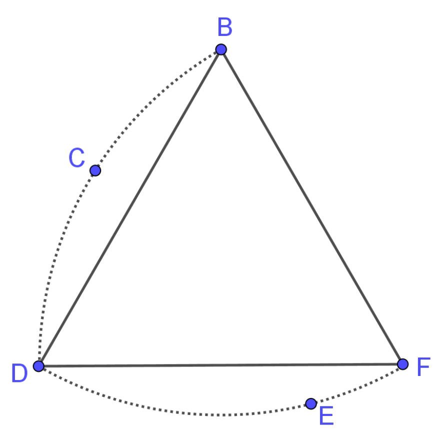
   (a) Show that it is possible for 7 of these diagonals to have the same length.

   (b) Show that it is impossible for 8 of these diagonals to have the same length.

Solution (a): Various constructions are possible. Perhaps the simplest approach is to start with a pair of equilateral triangles ACE and BDF of side length 1. Then we need only position the triangles so that the vertices ABCDEF are arranged in that order, so that D lies between C and E on the circle passing through A of radius 1. One possible arrangement is shown below:



Solution (b): Suppose by way of contradiction that ABCDEF had eight diagonals of length 1. Without loss of generality, suppose that the remaining diagonal has one endpoint at vertex A. Then diagonals BD, BE, BF, CE, CF, and DF all have length 1. Then triangle BDF is equilateral, and C lies on the 60-degree arc between B and D on the circle of radius 1 centered at F, and E lies on the 60-degree arc between D and F on the circle of radius 1 centered at B.



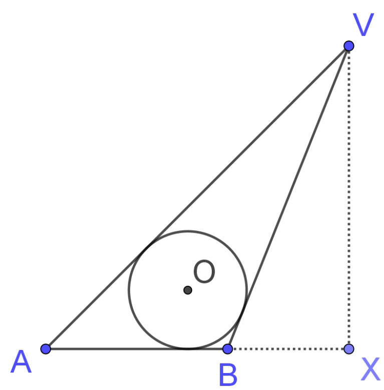
However, the distance from an interior point on one arc to any interior point of the other arc is always less than 1, which contradicts the assumption that CE has length 1. (For points C and E on these arcs, the distance CE can always be increased by moving C towards point B and moving E towards point F, but BF has length 1.) This is impossible, and so no such hexagon can exist.

Remark: The solution given to (b) shows that the two unequal diagonals for any example in (a) cannot share any endpoints. If these diagonals are BE and CF, then BDF and ACE are necessarily equilateral, and thus the example must have the form described in part (a).

5. A slanted cone (i.e., a cone such that the line between its vertex and the center of its base is not perpendicular to its base) has a base radius of 4cm and a volume of $128\pi\text{cm}^3$. The largest sphere that can be inscribed in this cone has volume $36\pi\text{cm}^3$. Compute the distance from the cone's vertex to the center of its base.

Answer: $2\sqrt{193}$ cm.

Solution: Suppose the vertex is V, the center of the sphere is O, and the radius of the sphere is r. Consider the plane $\mathcal{P}$ perpendicular to the plane containing the base of the cone that contains O and V. By symmetry, since the plane contains the center of the sphere, the intersection of $\mathcal{P}$ with the sphere is a great circle. This great circle is inscribed in the triangle ABV, where AB is a diameter of the cone. Because the cone is also symmetric across the plane $\mathcal{P}$, it must be the case that A is the point on the base farthest from V and B is the point on the base closest to V, yielding the diagram below:



Per the given information, the radius of the inscribed sphere is 3cm, which is also the radius of the inscribed circle of $\triangle ABV$, and the height of the cone is 24cm. If X is the foot of the perpendicular from V to AB, then if $AX = x$, we have $VX = 24$ and $XB = 8$. Then the area of $\triangle ABV$ is $K = \frac{1}{2} \cdot 8 \cdot 24 = 96$, and so since its inradius is $r = 3$, its perimeter must be $\frac{2K}{r} = 64$. By the Pythagorean theorem, we have $AV = \sqrt{x^2+576}$ and $BV = \sqrt{(x+8)^2+576}$, so the perimeter of $\triangle ABV$ is $\sqrt{x^2+576} + \sqrt{(x+8)^2+576} + 8 = 64$, so that $\sqrt{(x+8)^2+576} = 56 - \sqrt{x^2+576}$. Squaring both sides yields $(x+8)^2+576 = 56^2 - 112\sqrt{x^2+576} + x^2 + 576$, so that $7\sqrt{x^2+576} = 192 - x$. Squaring again yields $49(x^2+576) = 192^2 - 384x + x^2$ so that $48x^2 + 384x - 8640 = 0$, which factors as $48(x-10)(x+18) = 0$.

Thus, $x = 10$, and then $AO^2 = OX^2 + VX^2 = 14^2 + 12^2 = 4 \cdot 193$ and so $AO = \boxed{2\sqrt{193}\text{cm}}$.

6. A total of 3,432 students compete in the Very Mathematical Test Season (VMTS). The VMTS has 14 true/false problems, and each student answers every problem either correctly or incorrectly. Determine the greatest possible value of N such that the following statement must be true for any possible results of the competition:

   - There must necessarily exist a pair of 2 problems and a set of N students such that all of the N students answered both problems correctly or all of the N students answered both problems incorrectly.

   Answer: 792.

   Solution: If a student answers k questions correctly and $14 - k$ questions incorrectly, then they have $\binom{k}{2}$ pairs of correct answers and $\binom{14-k}{2}$ pairs of incorrect answers, for a total of $\binom{k}{2} + \binom{14-k}{2} = k(k-1)/2 + (24-k)(23-k)/2 = k^2 - 14k + 91 = (k-7)^2 + 42$. Therefore, the total number of paired responses for each student (either both correct or both incorrect) is at least 42, with equality if and only if every student correctly answers 7 problems. Summing over all students shows that the total number of paired responses, over all students, is at least $43 \cdot 3432$.

   We then have $\binom{14}{2} = 91$ possible pairs (i, j) of two questions, each of which can be answered either correctly or incorrectly. This yields 182 possible "buckets" (consisting of an unordered pair of two problems, along with either "correct" or "incorrect") into which each student's paired responses are placed. By the pigeonhole principle, since we have at least $43 \cdot 3432$ paired responses being placed into 182 buckets, some bucket must have at least $43 \cdot 3432/182 = 792$ objects placed into it. Each paired response in this bucket can come from only one student, so there are at least 792 students who all answered this pair of questions either correctly or incorrectly. Therefore, if $N = 792$, then the statement is necessarily true for any possible results of the competition.

   On the other hand, if $N = 792$, we could have equality everywhere above: this happens precisely when each student answers exactly 7 questions, and each of the 182 buckets has exactly 792 students in it (in other words, each pair of problems has exactly 792 students answer both correctly and exactly 792 students answer both incorrectly). This can be achieved when each of the $\binom{14}{7} = 3432$ students answers a different set of 7 questions: in that case, for any pair of problems there are exactly $\binom{12}{5} = 792$ students who answered both correctly and $\binom{12}{7} = 792$ students who answered both incorrectly. Therefore, the given condition is false in this scenario for any $N > 792$.

   We conclude that the maximum possible value of N is therefore $N = \boxed{792}$.