Vermont Mathematics Talent Search, Solutions to Test 4, 2021-2022

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. This is a relay problem. The answer to each part will be used in the next part.

   (a) Ms. Johnson writes 30 consecutive positive integers on her blackboard. In doing so, she writes a total of 13 zeroes, 43 ones, 13 twos, 29 threes, 3 fours, 3 fives, 3 sixes, 3 sevens, 3 eights, and 7 nines. What is the smallest of the integers Ms. Johnson wrote on her blackboard?

   Answer (a): 1296.

   Solution (a): There are a total of 120 digits written, so all of the integers must have been 4-digit numbers. Additionally, note that 30 consecutive integers will have each of the 10 possible units digits occur exactly 3 times: thus, discarding these units digits leaves 10 zeroes, 40 ones, 10 twos, 26 threes, and 4 nines. Since there are only 4 nines and 10 zeroes, it is not possible for the thousands digit to have changed (since each of the integers would necessarily contribute either a nine or a zero in the hundreds place), which means the thousands digit must be 1.
   Of the remaining 60 hundreds and tens digits, there are 4 nines, 10 zeroes, 10 ones, 10 twos, and 26 threes. The hundreds digits must consist of 30 total digits of two consecutive types, while the tens digits consist of 30 total digits with two runs of exactly 10 in the middle, bordered by two other types. Since the hundreds digits cannot all be the same, the nines must appear in the tens place, along with zeroes, ones, and twos, which leaves the 26 threes to appear in the hundreds place. Thus, the tens must have the 4 nines, 10 zeroes, 10 ones, and 6 twos, while the hundreds have 4 twos and 26 threes. Thus the smallest integer has thousands digit 1, hundreds digit 2, and is 4 less than a multiple of 100 (to produce the 4 nines in the tens digits): this means it is $\boxed{1296}$.

   (b) Let A be the answer to part (a). Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x have the property that $x + x^2 = \sqrt{A} + \lfloor x^2 \rfloor$?

   Answer (b): 72.

   Solution (b): Note that $\sqrt{A} = \sqrt{1296} = 36$. If we rearrange the equation as $x = 36 - (x^2 - \lfloor x^2 \rfloor)$, then since $0 \le x^2 - \lfloor x^2 \rfloor < 1$, we see that $35 < x \le 36$, and so x^2 ranges from 35^2 to 36^2. As x^2 increases through the interval $[35^2 + k, 35^2 + k + 1)$ for each $k = 0, 1, \dots, 70$, we see that $36 - (x^2 - \lfloor x^2 \rfloor)$ will decrease from 36 to 35, while x increases from $35 < \sqrt{35^2 + k}$ to $\sqrt{35^2 + k + 1} < 36$: thus, the graphs will intersect precisely once in this interval. Additionally, the graphs intersect when $x = 36$, for a total of $\boxed{72}$ values of x satisfying the given equation.

   (c) Let B be the answer to part (b). A hollow cone has total surface area $B\pi$ square centimeters. The circular base is cut off, and the remaining surface is cut once and unrolled into a B° circular sector. What is the area of the sector?

   Answer (c): 60π.

   Solution (c): Suppose the circular sector has radius ℓ, which is then also the slant height of the cone: then its area is $(B/360)\pi\ell^2$. Also, the circumference of the sector is $(B/360) \cdot 2\pi\ell$, which is also the circumference of the base of the cone. Therefore, if r is the cone's radius, we obtain $2\pi r = (B/360) \cdot 2\pi\ell$ so that $r = \ell B/360$ and thus the area of the base is $\pi r^2 = \pi\ell^2 B^2/360^2$. Since the area of the base plus the area of the sector is the total surface area $B\pi$, we obtain $(B/360) \cdot \pi\ell^2 + \pi\ell^2 B^2/360^2 = B\pi$, so that $\ell^2[1/360 + B/360^2] = 1$, so $\ell^2 = \dfrac{1}{1/360 + B/360^2}$. Then the area of the sector is $(B/360) \cdot \pi\ell^2 = \dfrac{360\pi B}{360 + B} = \boxed{60\pi}$ for $B = 72$.

2. Each vertex of an isosceles right triangle lies on one of the lines $y = 0$, $y = 2$, or $y = 6$. (One line may have two vertices while another has none.) What is the sum of all possible values of the area of the triangle?

Answer: 140.

Solution: Suppose the triangle is ABC with the right angle at A. We break into cases based on the possible lines containing the points:

- A and B lie on $y = 0$, C lies on $y = 2$, or the reverse: then the leg AC has length 2, so the area is 2.
- A and B lie on $y = 0$, C lies on $y = 6$, or the reverse: then the leg AC has length 6, so the area is 18.
- A and B lie on $y = 2$, C lies on $y = 6$, or the reverse: then the leg AC has length 4, so the area is 8.
- B and C lie on $y = 0$, A lies on $y = 2$, or the reverse: then the altitude to hypotenuse BC has length 2, so the area is 4.
- B and C lie on $y = 0$, A lies on $y = 6$, or the reverse: then the altitude to hypotenuse BC has length 6, so the area is 36.
- B and C lie on $y = 2$, A lies on $y = 6$, or the reverse: then the altitude to hypotenuse BC has length 4, so the area is 16.
- A lies on $y = 0$, B lies on $y = 2$, C lies on $y = 6$: then leg AB is the hypotenuse of a right triangle with legs 2 and 6, so $AB = \sqrt{2^2 + 6^2} = \sqrt{40}$, so the area is 20.
- A lies on $y = 2$, B lies on $y = 0$, C lies on $y = 6$: then leg AB is the hypotenuse of a right triangle with legs 2 and 4, so $AB = \sqrt{2^2 + 4^2} = \sqrt{20}$, so the area is 10.
- A lies on $y = 6$, B lies on $y = 0$, C lies on $y = 2$: then leg AB is the hypotenuse of a right triangle with legs 4 and 6, so $AB = \sqrt{4^2 + 6^2} = \sqrt{52}$, so the area is 26.

These cover all of the possible cases, so the possible areas are 2, 4, 8, 10, 16, 18, 20, 26, and 36, for a sum of $\boxed{140}$.

3. Find all prime numbers p such that $(p^2 + p + 6)^{p^3-2p^2+3} - (p-3)^{4p-8}$ is divisible by p.

Answer: 3, 5, 11, 53.

Solution: First note that $p = 2$ is not a solution since the expression reduces to $12^3 - (-1)^4$ which is odd, while $p = 3$ is a solution since the expression reduces to $18^{12} - 0^8$ which is divisible by 3. Now suppose $p > 3$: we are seeking solutions to the congruence $(p^2 + p + 6)^{p^3-2p^2+3} \equiv (p-3)^{4p-8} \pmod{p}$.

Note that $p \equiv 0 \pmod{p}$, so the given congruence is equivalent to $6^{p^3-2p^2+3} \equiv (-3)^{4p-8} \pmod{p}$. Additionally, recall that Fermat's little theorem says that $a^p \equiv a \pmod{p}$ for every integer a: then $a^{p^2} \equiv (a^p)^p \equiv a^p \equiv a \pmod{p}$ as well, and similarly we see $a^{p^3} \equiv a \pmod{p}$. Then, since 6 is relatively prime to p, we have $6^{p^3-2p^2+3} \equiv 6^{p^3}(6^{-1})^{2p^2}6^3 \equiv 6(6^{-1})^2 6^3 \equiv 6^2 \pmod{p}$. Similarly, we have $(-3)^{4p-8} \equiv (-3)^{4p}(-3)^{-8} \equiv (-3)^4(-3)^{-8} \equiv (-3)^{-4} \pmod{p}$.

Therefore, the original congruence is equivalent to $6^2 \equiv (-3)^{-4} \pmod{p}$, so multiplying both sides by $(-3)^4$ yields $6^2(-3)^4 \equiv 1 \pmod{p}$. This is in turn equivalent to saying that p divides $6^2(-3)^4 - 1 = 54^2 - 1 = 53 \cdot 55 = 5 \cdot 11 \cdot 53$, and so we must have $p = 5$, $p = 11$, or $p = 53$.

Since every step was an equivalence, we see that the full list of primes is $p = \boxed{3,\ 5,\ 11,\ 53}$.

4. The polynomial $p(x)$ has real coefficients and satisfies $p(x)^3 = x^{2022} + 20x^{2021} + 22x^{2020} + 2022x^{2019} + \cdots$, where the remaining terms have degree at most 2018. Suppose that the factorization of $p(x)$ over the real numbers is $p(x) = (x^2 + a_1x + b_1)(x^2 + a_2x + b_2)\cdots(x^2 + a_{337}x + b_{337})$ for real numbers $a_1, a_2, \ldots, a_{337}$ and $b_1, b_2, \ldots, b_{337}$. What is the value of the expression $\sum_{1\le i\le 337}(a_i + b_i) + \sum_{1\le j<k\le 337} a_ja_k$?

Answer: $-274/9$.

Solution: First, consider the product expansion of $p(x) = (x^2+a_1x+b_1)(x^2+a_2x+b_2)\cdots(x^2+a_{337}x+b_{337})$. The top-degree product is x^{674}, obtained by selecting x^2 from each term. The products corresponding to the power x^{673} require selecting an x from one term and x^2 from all remaining terms, so the coefficient of x^{673} is $a_1 + a_2 + \cdots + a_{337}$. The products corresponding to the power x^{672} require either selecting an x from two terms and x^2 from all remaining terms or selecting a 1 from one term and x^2 from all remaining terms, so the coefficient of x^{672} is $(a_1a_2 + a_1a_3 + \cdots + a_{336}a_{337}) + (b_1 + b_2 + \cdots + b_{337})$.

Therefore, we can see that the desired expression is simply the sum of the coefficients of x^{672} and x^{673} in $p(x)$. To compute these, note that by its factorization $p(x)$ must have leading coefficient 1, so suppose that $p(x) = x^{674} + cx^{673} + dx^{672} + \cdots$. Then $p(x)^3 = x^{2022} + 3cx^{2021} + (3c^2 + 3d)x^{2020} + \cdots$ by a similar calculation to the one above. Comparing coefficients yields $3c = 20$ and $3c^2 + 3d = 22$, so that $c = 20/3$ and then $d = -334/9$. Then the desired sum is $c + d = \boxed{-274/9}$.

Remark: The coefficient of x^{2019} does not affect the answer to the problem, and was included only to make it more difficult to obtain the answer by directly finding an example of a factorization $p(x)$.

5. Evan has a deck of 27 cards that are numbered 1 through 27 inclusive, where each number appears on exactly one card. He shuffles the cards, deals them from the deck in sets of 3, and then from each set he keeps the middle-valued card and discards the highest and lowest values, resulting in a smaller set of 9 cards. He repeats this process, shuffling the cards, dealing them in sets of 3, and keeping the middle-valued card from each set while discarding the others, to yield 3 cards. He finally selects the middle-valued card from this set of 3. Suppose the value on this middle card is M.

(a) What is the minimum possible value of M?

(b) What is the probability that M equals its minimum possible value?

Answers: (a) 8 (b) 81/82225.

Solution (a): We claim the minimum possible value of M is 8. To see this first observe that $M = 8$ is achievable as follows: start with initial triples $\{1, 2, 9\}$, $\{3, 4, 10\}$, $\{5, 6, 11\}$, $\{7, 8, 12\}$, $\{13, 14, 15\}$, $\{16, 17, 18\}$, $\{19, 20, 21\}$, $\{22, 23, 24\}$, $\{25, 26, 27\}$. The nine remaining middle cards are 2, 4, 6, 8, 14, 17, 20, 23, and 26. If they are then grouped $\{2, 4, 14\}$, $\{6, 8, 17\}$, $\{20, 23, 26\}$, the three middle cards left are 4, 8, 23, and then the last remaining card is 8.

To see that $M \ge 8$, suppose the final three cards are $m_1 < m_2 < m_3$, where $M = m_2$. Then of the final nine cards, since m_1 and m_2 are the middle values in their respective triples, there must be two more values $m_3 < m_1$ and $m_4 < m_2$ among these nine cards. Finally, since m_1, m_2, m_3, m_4 themselves are medians from a set of 3 cards, there must be four additional values $m_5 < m_1$, $m_6 < m_2$, $m_7 < m_3$, and $m_8 < m_4$. Then all eight values $m_1, \ldots, m_8$ are less than or equal to M, and so $M \ge 8$.

Solution (b): Using the analysis in (a), we see that we can only have $M = 8$ when after the first deal we have four of the numbers 1-8 included in our remaining medians, and after the second deal we have two of 1-8 still included. To find the probability that this occurs, color the cards 1-8 red and the others blue. For the first deal, we are arranging 8 red cards and 19 blue cards into 9 triples, and need to find the probability that there are 4 triples each of which has a pair of red cards. There are $\binom{9}{4}$ ways to select the triples which have a pair of red cards, and 3^4 ways to select the cards in each of these triples that are red. Since in total there are $\binom{27}{8}$ ways to select the locations of the red cards, the probability here is $\binom{9}{4}3^4/\binom{27}{8}$. After this first deal, we are arranging 4 red cards and 5 blue cards into 3 triples, and need the probability that 2 triples each have a pair of red cards. By the same sort of analysis, the probability is $\binom{3}{2}3^2/\binom{9}{4}$. After this deal we are guaranteed that the last card remaining is labeled 8. Therefore the total probability that $M = 8$ is $\frac{\binom{9}{4}3^4}{\binom{27}{8}} \cdot \frac{\binom{3}{2}3^2}{\binom{9}{4}} = \frac{\binom{3}{2}3^6}{\binom{27}{8}} = \boxed{\frac{81}{82225}}$.

6. Kiran and Evan are playing a number-guessing game. Evan chooses a 3-digit sequence 000 through 999 inclusive. Kiran then makes a sequence of 3-digit guesses, and after each guess, Evan tells him either "you had at least one correct digit" or "none of your digits was correct". For example, if Evan's number is 382 and Kiran guesses 311 or 182 or 382, Evan will say that at least one digit was correct, whereas if Kiran guesses 014, 123, or 238, Evan will say that none of the digits was correct.

   (a) Prove that Kiran, if he uses an optimal strategy, can guarantee that he knows Evan's number after he has made 13 guesses.

   (b) Prove that if Kiran only has 12 guesses, then he cannot necessarily determine Evan's number definitively.

Solution (a): We describe a strategy by which Kiran can determine Evan's number with at most 13 guesses. Kiran starts by guessing 000, 111, ... , 999 (a total of 10 guesses) and keeping track of which of these yield an answer of at least one correct digit. There are then three cases:

- Only one guess results in a correct digit. Then none of the other digits appear in the sequence, so the sequence is ddd where d was the digit.
- Two guesses result in a correct digit. By relabeling, suppose the correct digits were 8 and 9. Then the number is one of 889, 898, 899, 988, 989, 998. Kiran then guesses "900", "090", and "009". Each "yes" answer corresponds to a 9 in the corresponding place, and since Kiran knows there are no 0s in the number, so Kiran can determine the number from these three responses.
- Three guesses result in a correct digit. By relabeling, suppose the correct digits are 7, 8, and 9. Then the number is one of 789, 798, 879, 897, 978, 987. Kiran then guesses "900", "090". If the first answer is "yes", then the sequence is either 987 or 978 and Kiran guesses "080" to distinguish them. If the second answer is "yes" then the sequence is either 897 or 798 and Kiran guesses "800" to distinguish them. If both answers are "no" then the sequence is either 879 or 789 and Kiran guesses "800" to distinguish them.

In all cases, Kiran can determine Evan's number in 13 guesses.

Solution (b): First note that making k guesses can distinguish among at most 2^k possible sequences by the pigeonhole principle, so if there are more than 2^k remaining sequences, then at least $k+1$ additional guesses will be required.

Suppose Kiran makes his first 6 guesses. At most 6 possibilities have been tried for each digit, and so if the answer to all 6 guesses is "no correct digits", there are at least 4 possibilities for each of the digits and thus $4 \cdot 4 \cdot 4 = 64$ possibilities for the sequence. This is nearly enough: since $64 = 2^6$, our observation above implies that at least 6 more guesses are needed.

We can get the necessary improvement by noting that regardless of Kiran's 7th guess, he actually needs at least six more:

- If he uses no new numbers, he gains no new information.
- If he uses a new number in 1 digit, then if the answer is "no correct digits", there are $3 \cdot 4 \cdot 4 = 48$ remaining possibilities.
- If he uses new numbers in 2 digits, then if the answer is "no correct digits", there are $3 \cdot 3 \cdot 4 = 36$ remaining possibilities.
- If he uses new numbers in all 3 digits, then if the answer is "at least one correct digit", there are $64 - 3 \cdot 3 \cdot 3 = 37$ remaining possibilities.

In each of these situations, at least 6 additional guesses are needed after the first 7, and so Kiran cannot guarantee that he can determine Evan's number after only 12 guesses.